On a Recent Conjecture by Z. Van Herstraeten and N. J. Cerf for the Quantum Wigner Entropy

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- Introduction: Quantum entropies
- Quantum Wigner entropy; Feichtinger states
- A new Lieb-type inequality for Wigner functions
- Proof that the quantum Wigner entropy is bounded below
- Wigner positivity

- Density matrix:

$$\widehat{\rho} = \sum_j p_j |\psi_j >< \psi_j| \ ,$$

with $0 \leq p_j \leq 1$, $\sum_j p_j = 1$, $< \psi_j | \psi_k >= \delta_{j,k}$.

- von Neumann entropy:

$$S_{vN}\left[\widehat{\rho}\right] = -Tr\left(\widehat{\rho} ln \widehat{\rho}\right) = -\sum_j p_j \ln p_j \ .$$

- Positivity: $S_{vN}\left[\widehat{\rho}\right] \geq 0$, with $S_{vN}\left[\widehat{\rho}\right] = 0$ if and only if $\widehat{\rho} = |\psi >< \psi|$ is a pure state.

For $\psi \in L^2(\mathbb{R}^n)$ and $\|\psi\|_{L^2} = 1$, $|\psi(x)|^2$ is the probability density for the particle's position. Define its Shannon entropy:

$$H\left[|\psi|^2\right] := -\int_{\mathbb{R}^n} |\psi(x)|^2 \ln\left(|\psi(x)|^2\right) dx \ .$$

Not bounded below! Example: for the Gaussian measure

$$\psi(x) = \sqrt[4]{\frac{1}{\pi a^2}} e^{-\frac{x^2}{2a^2}} \ , \ a > 0$$

we have:

$$H\left[|\psi|^2\right] = \frac{1}{2} \ln(\pi a^2 e) \ .$$

Thus, as $a \to 0^+$, we have $H\left[|\psi|^2\right] \to -\infty$.

However, if we consider also the Shannon entropy of the Fourier transform

$$(\mathcal{F}_\hbar \psi)(p) := \frac{1}{(2\pi\hbar)^{n/2}} \int_{\mathbb{R}^n} \psi(x) e^{-\frac{i}{\hbar} x \cdot p} dx \ ,$$

we obtain the entropic uncertainty principle (Hirshman (1957), Beckner (1975), Biclynicki-Birula and Mycielski (1975)):

Theorem

$$H\left[|\psi|^2\right] + H\left[|\mathcal{F}_\hbar \psi|^2\right] \geq n \ln(\pi \hbar e) \ ,$$

with equality if and only if ψ is a Gaussian.

This is stronger than Heisenberg UP. In dimension $n = 1$ Shannon's inequality reads:

$$H(\mu) \leq \ln \sqrt{2\pi e V(\mu)} \ ,$$

where $V(\mu)$ is the variance associated with the probability measure μ. Then, it follows that:

$$\left(\frac{\hbar}{2}\right)^2 \leq \frac{e^{2H\left[|\psi|^2\right]+2H\left[|\mathcal{F}_\hbar\psi|^2\right]}}{(2\pi e)^2} \leq V(|\psi|^2)V(|\mathcal{F}_\hbar\psi|^2)$$

- Rényi entropy of μ:

$$H_\alpha [\mu] := \frac{\alpha}{1-\alpha} \ln \left(\|\mu\|_\alpha \right)$$

for $1 < \alpha \leq \infty$.

$$\lim_{\alpha \to 1} H_\alpha [\mu] = H [\mu]$$

- Entropic uncertainty principle:

Theorem

$$H_\alpha\left[|\psi|^2\right] + H_\beta\left[|\mathcal{F}_\hbar\psi|^2\right] \geq n\left[\ln(2\pi\hbar) + \frac{\ln(2\alpha)}{2(\alpha-1)} + +\frac{\ln(2\beta)}{2(\beta-1)}\right]$$

provided 2α *and* 2β *are Hölder conjugates:*

$$\frac{1}{\alpha} + \frac{1}{\beta} = 2$$

- If $\alpha \to 1$ we recover the Hirshmann UP.

- Let $S \in Sp(2n)$ be a free symplectic matrix:

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \ \det B \neq 0$$

- Metaplectic operators:

$$\left(Mp(S)\psi\right)(\zeta) = \left(\widehat{S}\psi\right)(\zeta) :=$$

$$\frac{1}{(2\pi\hbar i)^{n/2}\sqrt{|\det B|}} \int_{\mathbb{R}^n} f(x) e^{\frac{i}{2\hbar}(\zeta \cdot DB^{-1}\zeta + xB^{-1}Ax) - \frac{i}{\hbar} x \cdot B^{-1}\zeta} dx$$

Examples:

- **Fourier transform:** $S = J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$
- **Fractional Fourier transform:**
  $A = D = \text{diag}(\cos(\theta_1), \cdots, \cos(\theta_n))$ and
  $B = -C = \text{diag}(\sin(\theta_1), \cdots, \sin(\theta_n))$
- **Fresnel transform:** $A = D = I$, $C = 0$ and
  $B = \text{diag}(b_1, \cdots, b_n)$
- **Lorentz transform:** $A = D = \text{diag}(\cosh(\phi_1), \cdots, \cosh(\phi_n))$
  and $B = C = \text{diag}(\sinh(\phi_1), \cdots, \sin(\phi_n))$

- Entropic uncertainty principle:

Theorem

$$H\left[|\widehat{S_1}\psi|^2\right] + H\left[|\widehat{S_2}\psi|^2\right] \geq n \ln\left[\det(B_2A_1^T - A_2B_1^T)\pi e\hbar\right]$$

- Phase space: $\mathbb{R}^n \times (\mathbb{R}^n)^* \simeq \mathbb{R}^{2n}$, $z = (x, p) \in \mathbb{R}^{2n}$
- Wigner functions:

$$|\psi><\phi| \mapsto \quad W(\psi,\phi)(x,p) = \\ = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} \psi(x+y/2)\overline{\phi(x-y/2)} e^{-\frac{i}{\hbar} y\cdot p} dx$$

$$|\psi><\psi| \mapsto \quad W\psi(x,p) = \\ = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} \psi(x+y/2)\overline{\psi(x-y/2)} e^{-\frac{i}{\hbar} y\cdot p} dx$$

$$\left(\widehat{\rho}\phi\right)(x) = \int_{\mathbb{R}^n} \rho(x,y)\phi(y)dy$$

where

$$\rho(x,y) = \sum_j p_j \psi_j(x)\overline{\psi_j(y)}$$

Then:

$$\begin{aligned} \widehat{\rho} \mapsto \quad & W\rho(x,p) = \\ & = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} \rho(x+y/2, x-y/2) e^{-\frac{i}{\hbar} y\cdot p} dx = \\ & = \sum_j p_j W\psi_j(x,p) \end{aligned}$$

- Husimi function:

$$\widehat{\rho} \mapsto = Q\rho(z) = \frac{1}{(\pi\hbar)^n} \int_{\mathbb{R}^{2n}} e^{-\frac{1}{\hbar}|z-\omega|^2} W\rho(\omega) d\omega$$

$$Q\rho(z) \geq 0 \ , \ \forall z \in \mathbb{R}^{2n}$$

- Wehrl entropy:

$$S_W[\rho] = H[Q\rho] = -\int_{\mathbb{R}^{2n}} Q\rho(z) \ln(Q\rho(z)) \, dz$$

- Entropic UP:

Theorem (Lieb,Carlen)

We have:

$$S_W[\rho] \geq 1$$

Moreover, we have an equality if and only if $\widehat{\rho}$ is a Gaussian coherent state.

- Lieb entropy:

$$S_L[\psi] = H\left[\frac{|W\psi|^2}{\|W\psi\|_2^2}\right] = -\int_{\mathbb{R}^{2n}} \frac{|W\psi(z)|^2}{\|W\psi\|_2^2} \ln\left(\frac{|W\psi(z)|^2}{\|W\psi\|_2^2}\right) dz$$

- Entropic UP:

Theorem (Lieb)

We have:

$$S_L[\psi] \geq 2n \ln\left(\frac{\pi\hbar e}{2}\right)$$

Moreover, we have an equality if and only if ψ is a Gaussian function.

- Problem: Not easy to extend to mixed states!

- Quantum Wigner entropy suggested by van Herstraeten and Cerf:

$$S_{HC}[\rho] = H[W\rho] = -\int_{\mathbb{R}^{2n}} W\rho(z) \ln\left(W\rho(z)\right) dz$$

- Problem: Wigner functions are not positive in general (Hudson's Theorem). Thus, the definition of quantum Wigner entropy applies to **Wigner-positive states** $W\rho \in \mathcal{W}^{+}$ only!

Conjecture (van Herstraeten, Cerf)

1. *There exists $C > -\infty$, such that*

$$S_{HC}[\rho] \geq C \ , \ \forall W\rho \in \mathcal{W}^+$$

2. *The sharp constant is: $C = n \ln(e\pi\hbar)$*
3. *The minimum is attained in $\mathcal{W}^+$ if and only if $W\rho$ is a Gaussian pure state.*

Valid for:

- Gaussian states
- Passive phase-invariant states of the harmonic oscillator:

$$Tr\left[\widehat{\rho}_p\widehat{H}\right]\leq Tr\left[\widehat{U}\widehat{\rho}_p\widehat{U}^{\dagger}\widehat{H}\right]$$

Harmonic oscillator (W. Pusz and S.L. Woronowicz, 1978):

$$\widehat{\rho}_p=\sum_{k=0}^{\infty}p_k|k><k|\ ,\ p_k\geq p_{k+1}$$

- Rényi entropy:

$$S_\alpha[\rho] = H_\alpha[W\rho] = \frac{\alpha}{1-\alpha}\ln\left(\|W\rho\|_\alpha\right)\ ,\ 1 < \alpha \leq \infty$$

Conjecture (van Herstraeten, Cerf)

We have

$$S_\alpha[\rho] \geq n\left(\ln(\pi\hbar) + \frac{\ln(\alpha)}{\alpha - 1}\right)\ ,$$

for all $W\rho \in \mathcal{W}^+$.

- Our results (N.C. Dias, JNP, 2023):
  Let $W\rho \in L^1(\mathbb{R}^{2n})$ and define

$$\mu_\rho(x,p) = \frac{|W\rho(x,p)|}{\|W\rho\|_{L^1}}$$

Theorem (Dias, JNP, 2023)

We have

$$H[\mu_\rho] \geq n \ln(2\pi\hbar) \ ,$$

for all $W\rho \in L^1(\mathbb{R}^{2n})$.

- $|C_{vHC} - C_{DP}| = n\,(ln(e\pi\hbar) - ln(2\pi\hbar)) \simeq 0.306853n$

Theorem (Dias,JNP)

Let $W\rho \in L^1(\mathbb{R}^{2n})$. We have:

- *For $2 \leq \alpha \leq \infty$:*

$$H_\alpha\left[\mu_\rho\right] \geq n\left(\ln(\pi\hbar) + \frac{\ln(\alpha)}{\alpha - 1}\right) ,$$

  with an equality if and only if $W\rho$ is a Gaussian pure state.

- *For $1 < \alpha < 2$:*

$$H_\alpha\left[\mu_\rho\right] \geq n\ln(2\pi\hbar) ,$$

  and there is no $W\rho$ for which we have an equality.

- Notice that $0.693147 \simeq \ln(2) < \frac{\ln(\alpha)}{\alpha-1} < 1$, for all $1 < \alpha < 2$.

- Lieb-type inequality:

Theorem (Dias, JNP, 2023)

Let q, θ be such that $1 < q < 2$ and $2 - q < \theta < 1$. Suppose that $W(f, g) \in L^1(\mathbb{R}^{2n})$ and $f, g \in L^2(\mathbb{R}^n)$. Then:

$$\|W(f,g)\|_{L^q} \leq \left(\frac{\pi\hbar}{2}\right)^{n/q} \left[\frac{2(1-\theta)}{\pi\hbar(q-\theta)}\right]^n \times$$

$$\times \left(2^n \|W(f,g)\|_{L^1}\right)^{\theta/q} \|f\|_{L^2} \|g\|_{L^2}$$

Corollary (Dias, JNP, 2023)

With the same conditions and $W\rho \in L^1(\mathbb{R}^{2n})$:

$$\|W\rho\|_{L^q} \leq \frac{1}{(\pi\hbar)^{n/q'}} \left(\frac{1-\theta}{q-\theta}\right)^{\frac{n}{q}(1-\theta)} \|W\rho\|_{L^1}$$

- Let

$$\mu_\rho(z) := \frac{|W\rho(z)|}{\|W\rho\|_{L^1}}$$

- We define the **quantum Wigner entropy** of $\widehat{\rho}$ as:

$$S_{HC}[\rho] = H[\mu_\rho] = -\int_{\mathbb{R}^{2n}} \mu_\rho(z) \ln(\mu_\rho(z))\, dz$$

- Properties:
  (1) For Wigner-positive states, $\mu_\rho = W\rho$, and $H[\mu_\rho] = H[W\rho]$.
  (2) For $\widehat{\rho} = \widehat{\rho_A} \otimes \widehat{\rho_B}$, we have $\mu_\rho = \mu_{\rho_A} \cdot \mu_{\rho_B}$ and hence:

$$H[\mu_\rho] = H[\mu_{\rho_A}] + H[\mu_{\rho_B}]$$

- Properties (cont.):
  (3) Let $S \in Sp(2n)$, $\widehat{S}$ an associated metaplectic operator, and $\widehat{\rho}$ a density matrix. Let

$$\widehat{\rho}' = \widehat{S}\widehat{\rho}\widehat{S}^{-1}$$

If the associated Wigner functions are $W\rho$ and $W\rho'$, then:

$$W\rho'(z) = W\rho(S^{-1}z)$$

Consequently:

$$H\left[\mu_{\rho'}\right] = H\left[\mu_{\rho}\right]$$

Quantum Wigner entropy; Feichtinger states

- Properties (cont.):
  (4) $\mu_\rho(x,p)$ does not have the correct marginals (except for Wigner-positive states).
  For example: 1st excited state of the harmonic oscillator

$$h_1(x) = \frac{1}{\sqrt[4]{\pi\hbar}}\sqrt{\frac{2}{\hbar}}\, x e^{-\frac{x^2}{2\hbar}}$$

with Wigner function

$$Wh_1(z) = \frac{1}{\pi\hbar}\left(\frac{2z^2}{\hbar} - 1\right) e^{-\frac{z^2}{\hbar}}$$

and

$$\mu_1(z) = \frac{e^{-\frac{z^2}{\hbar}}}{\pi\hbar\left(4/\sqrt{e} - 1\right)}\left|\frac{2z^2}{\hbar} - 1\right|$$

Quantum Wigner entropy; Feichtinger states

- Properties (cont.):

  (4) If we consider $|x| \geq \sqrt{\frac{\hbar}{2}}$, then:

$$\int_{\mathbb{R}} \mu_1(x,p)dp = \frac{x^2 e^{-\frac{x^2}{\hbar}}}{\hbar\sqrt{\pi\hbar}\left(1/\sqrt{e} - 1/4\right)}$$

  which is different from

$$|h_1(x)|^2 = \frac{2x^2}{\hbar\sqrt{\pi\hbar}} e^{-\frac{x^2}{\hbar}}$$

  (5) The quantum Wigner entropy is, in general, not concave (except for convex combinations of Wigner-positive states)

- The domain of the quantum Wigner entropy
  With our definition, we require $W\rho \in L^1(\mathbb{R}^{2n})$. There are Wigner functions for which this is not true. A simple example is the free particle in a box $0 \leq x \leq 1$ (in units $\hbar = 1$):
  $$\psi_n(x) = \sqrt{2}\sin(n\pi x)\ ,\ \ n \in \mathbb{N}$$
  with Wigner function
  $$W\psi_n(x,p) = \frac{2}{\pi}\left[\frac{\sin(2x(p-n\pi))}{4(p-n\pi)} + \right.$$
  $$\left. + \frac{\sin(2x(p+n\pi))}{4(p+n\pi)} - \cos(2n\pi x)\frac{\sin(2px)}{2}\right]$$
  Clearly, $W\psi_n \notin L^1(\mathbb{R}^{2n})$.

- The domain of the quantum Wigner entropy (cont.)
  We call states such that $W\rho \in L^1(\mathbb{R}^{2n})$ **Feichtinger states** (see C. de Gosson, M. de Gosson, 2021).
  This is by analogy with the Feichtinger algebra $S_0(\mathbb{R}^n)$.
  We have $\psi \in S_0(\mathbb{R}^n)$ if $W\psi(z) \in L^1(\mathbb{R}^{2n})$. This is a Banach space with respect to the norm

$$\|\psi\|_{S_0} := \|W(\psi, \phi)\|_{L^1}$$

  for some fixed $\phi \in \mathcal{S}(\mathbb{R}^n)$. It is a **minimal space** in the sense that it is the smallest Banach space which is invariant under phase-space translations. Moreover, it is a Banach algebra with respect to both pointwise multiplication and convolution (H.G. Feichtinger, 1981).

A new Lieb-type inequality for Wigner functions

- Lieb inequalities:

Theorem (Lieb, 1990)

Let $q \geq 2$ and $q' \leq p, p' \leq q$. If $f \in L^p(\mathbb{R}^n)$ and $g \in L^{p'}(\mathbb{R}^n)$, then $W(f,g) \in L^q(\mathbb{R}^{2n})$ and we have:

$$\|W(f,g)\|_{L^q(\mathbb{R}^{2n})} \leq C(p,q,n)\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^{p'}(\mathbb{R}^n)} \ ,$$

where $C(p,q,n)$ is a positive constant. For $q=2$, we have an equality for all $f,g \in L^2(\mathbb{R}^n)$. If $q>2$, we have an equality if and only if f and g are a matched Gaussian pair.

A new Lieb-type inequality for Wigner functions

- Lieb inequalities (cont.): A straightforward corollary is:

Corollary (N.C. Dias, JNP, 2023)

Let $W\rho$ be some Wigner function and $2 \leq q < \infty$. Then we have:

$$||W\rho||_{L^q(\mathbb{R}^{2n})} \leq \left[\frac{1}{q(\pi\hbar)^{q-1}}\right]^{n/q}$$

If $q = 2$, we have an equality if and only if the state is pure. If $q > 2$, we have an equality if and only if $W\rho$ is a Gaussian pure state.

A new Lieb-type inequality for Wigner functions

An immediate consequence is a simple proof that the Wigner-Rényi entropy

$$S_\alpha[\rho] = \frac{\alpha}{1-\alpha} \ln\left(\|\mu_\rho\|_\alpha\right)$$

is bounded below for $\alpha \geq 2$.
Since $\|W\rho\|_{L^1} \geq 1$, we have:

$$\|\mu_\rho\|_\alpha = \frac{\|W\rho\|_{L^\alpha}}{\|W\rho\|_{L^1}} \leq \|W\rho\|_{L^\alpha} \leq \left[\frac{1}{\alpha(\pi\hbar)^{\alpha-1}}\right]^{n/\alpha}$$

Thus:

$$S_\alpha[\rho] \geq n\left[\ln(\pi\hbar) + \frac{\ln(\alpha)}{\alpha-1}\right]$$

with an equality if and only if $W\rho$ is a Gaussian pure state. This proves the conjecture of van Herstraeten and Cerf for the case $\alpha \geq 2$.

A new Lieb-type inequality for Wigner functions

However, for the case $1 < \alpha < 2$, the situation is more delicate. There is a Lieb inequality but with an opposite inequality:

- Lieb inequalities (cont.):

Theorem (Lieb, 1990)

Suppose that $y \mapsto f(x + y/2)\overline{g(x - y/2)} \in L^1(\mathbb{R}^n)$ *for a.e. fixed* $x \in \mathbb{R}^n$, *and* $0 < \|W(f,g)\|_{L^q(\mathbb{R}^{2n})}$ *for some* $1 \leq q < 2$. *Then* $f \in L^p(\mathbb{R}^n)$ *and* $g \in L^{p'}(\mathbb{R}^n)$ *for* $q \leq p, p' \leq q'$, *and we have:*

$$\|W(f,g)\|_{L^q(\mathbb{R}^{2n})} \geq C(p,q,n)\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^{p'}(\mathbb{R}^n)} ,$$

where $C(p,q,n)$ *is a positive constant. Moreover, we have an equality if and only if* f *and* g *are a matched Gaussian pair.*

A new Lieb-type inequality for Wigner functions

We need a new inequality:

Theorem (Dias, JNP, 2023)

Let q, θ be such that $1 < q < 2$ and $2 - q < \theta < 1$. Suppose that $W(f,g) \in L^1(\mathbb{R}^{2n})$ and $f, g \in L^2(\mathbb{R}^n)$. Then:

$$\|W(f,g)\|_{L^q} \leq \left(\frac{\pi\hbar}{2}\right)^{n/q} \left[\frac{2(1-\theta)}{\pi\hbar(q-\theta)}\right]^n \times$$

$$\times \left(2^n \|W(f,g)\|_{L^1}\right)^{\theta/q} \|f\|_{L^2} \|g\|_{L^2}$$

A new Lieb-type inequality for Wigner functions

Corollary (Dias, JNP, 2023)

With the same conditions and $W\rho \in L^1(\mathbb{R}^{2n})$*:*

$$\|W\rho\|_{L^q} \leq \frac{1}{(\pi\hbar)^{n/q'}} \left(\frac{1-\theta}{q-\theta}\right)^{\frac{n}{q}(1-\theta)} \|W\rho\|_{L^1}$$

Proof that the quantum Wigner entropy is bounded below

Theorem (Dias, JNP, 2023)

We have

$$S_{HC}[\rho] = H[\mu_\rho] \geq n \ln(2\pi\hbar) ,$$

for all Feichtinger states $W\rho \in L^1(\mathbb{R}^{2n})$. Moreover, there is no $W\rho$ for which we have an equality.

- Sketch of the proof: Let

$$\theta = 1 - \epsilon , \ q = 1 + \epsilon + \epsilon^2$$

where $\epsilon > 0$ is sufficiently small, so that:

$$0 < 2 - q < \theta < 1$$

Proof that the quantum Wigner entropy is bounded below

- Sketch of the proof (cont.): Let

$$K_\epsilon := \frac{1}{\epsilon+\epsilon^2}\left(J_\rho(1) - J_\rho(1+\epsilon+\epsilon^2)\right)$$

where $J_\rho(q)$ denotes the functional

$$J_\rho(q) := \|(\pi\hbar)^n \mu_\rho\|_{L^q}^q$$

Since $J_\rho(1) = (\pi\hbar)^n$ we have from our inequality:

$$K_\epsilon > \frac{1}{\epsilon+\epsilon^2}\left((\pi\hbar)^n - \frac{(\pi\hbar)^n}{(2+\epsilon)^{n\epsilon}}\right) =$$

$$= \frac{(\pi\hbar)^n}{\epsilon+\epsilon^2}\left(1 - \frac{1}{(2+\epsilon)^{n\epsilon}}\right) = (\pi\hbar)^n n \ln(2) + \mathcal{O}(\epsilon)$$

Proof that the quantum Wigner entropy is bounded below

- Sketch of the proof (cont.):
  Now consider the inequality:

$$0 \leq \frac{1}{\sigma} X(1 - X^{\sigma}) \leq -X \ln(X) ,$$

which holds for $\sigma > 0$ and all $X \in [0, 1]$. If we set $\sigma = \epsilon + \epsilon^2$ and $X = (\pi\hbar)^n \mu_\rho(z)$, then:

$$0 \leq \frac{(\pi\hbar)^n \mu_\rho(z)}{\epsilon+\epsilon^2} \left[1 - \left((\pi\hbar)^n \mu_\rho(z) \right)^{\epsilon+\epsilon^2} \right] \leq$$

$$\leq -(\pi\hbar)^n \mu_\rho(z) \ln\left((\pi\hbar)^n \mu_\rho(z) \right)$$

Proof that the quantum Wigner entropy is bounded below

- Sketch of the proof (cont.): Upon integration over $z \in \mathbb{R}^{2n}$ we obtain:

$$0 \leq K_\epsilon \leq (\pi\hbar)^n \left(H\left[\mu_\rho\right] - n \ln(\pi\hbar)\right)$$

Taking the limit $\epsilon \to 0$, we obtain:

$$H\left[\mu_\rho\right] \geq n \ln(2\pi\hbar)$$

- Epilogue: A discussion about Wigner-positive states

Theorem (Hudson, 1974; Soto and Claverie, 1983)

The Wigner function of a pure state

$$W\psi(x,p) = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} \psi(x+y/2)\overline{\psi(x-y/2)} e^{-\frac{i}{\hbar} y\cdot p} dy$$

is everywhere non-negative if and only if ψ *is a generalized Gaussian:*

$$\psi(x) = e^{-x\cdot Ax + b\cdot x + c}$$

where $A \in \mathbb{C}^{n\times n}$, $Re(A) > 0$, $b \in \mathbb{C}^n$ *and* $c \in \mathbb{C}$.

For mixed states the situation is much more complicated.

- Various constructions of Wigner-positive states:

(I) A **Gaussian measure**

$$F(z) = \frac{e^{-\frac{1}{2}(z-z_0)\cdot\Sigma^{-1}(z-z_0)}}{(2\pi)^n\sqrt{\det\Sigma}} \ , \ z \in \mathbb{R}^{2n}$$

is the Wigner function of some density matrix if and only if its covariance matrix Σ satisfies the Robertson-Schrödinger uncertainty principle:

$$\Sigma + \frac{i\hbar}{2} J \geq 0$$

(II) Since $\mathcal{W}^+$ is a convex set, we can obtain new Wigner-positive states through **convex combinations** of other known Wigner-positive states.

(III) One can also obtain Wigner-positve states through suitable **convolutions.** A notable example is the Husimi function: it is the convolution of a Gaussian with some Wigner function which is also a Wigner function and everywhere non-negative.

(III) (Cont.) Let $\mathcal{S}\left(L^2(\mathbb{R}^n)\right)$ be the set of density matrices. Through the Weyl map it is isomorphic with the set of Wigner functions $\mathcal{W}_\hbar(\mathbb{R}^{2n})$:

$$W_\hbar : \begin{array}{ccc} \mathcal{S}\left(L^2(\mathbb{R}^n)\right) & \to & \mathcal{W}_\hbar(\mathbb{R}^{2n}) \\ \widehat{\rho} & \mapsto & W\rho(z) \end{array}$$

Let us replace $\hbar$ by some other constant $\alpha \in \mathbb{R}$.

Definition

Let $F \in L^2(\mathbb{R}^{2n}) \cap C(\mathbb{R}^{2n})$ be real and normalized. We say that F is of the α-positive type if $W_\alpha^{-1}(F) \in \mathcal{S}\left(L^2(\mathbb{R}^n)\right)$. The **Narcowich-Wigner spectrum** of F is:

$$\mathrm{Spec}(F) := \{\alpha \in \mathbb{R} : \ F \text{ is of the } \alpha\text{-positive type } \}$$

Theorem (Kastler, Loupias, Miracle-Sole)

Let $F \in L^2(\mathbb{R}^{2n})$ and denote by $\widehat{F}$ its symplectic Fourier transform:

$$\widehat{F}(a) = (\mathcal{F}_\sigma F)(a) := \int_{\mathbb{R}^{2n}} F(z) e^{i\sigma(z,a)} dz$$

Then F is of the α-positive type if and only $\widehat{F}$ is continuous and the $m \times m$ matrix with entries

$$M_{jk} = \widehat{F}(a_j - a_k) e^{\frac{i\alpha}{2}\sigma(a_j, a_k)}$$

is hermitean and non-negative for any $m \in \mathbb{N}$, and any set of m points $a_1, \cdots, a_m \in \mathbb{R}^{2n}$.

- Bochner's theorem for positive-definite functions: $\alpha = 0$

Properties:

- $\alpha \in \text{Spec}(F) \Leftrightarrow -\alpha \in \text{Spec}(F)$
- $\text{Spec}(F) = \text{Spec}(F \circ S)$, for any $S \in Sp(2n)$.
- If F is a Gaussian state, then $\text{Spec}(F) = [-\lambda_{max}, \lambda_{max}]$, where λ_{max} is largest symplectic eigenvalue of the covariance matrix of F.
- If $\alpha \in \text{Spec}(F)$ and $\beta \in \text{Spec}(G)$, then $\alpha + \beta \in \text{Spec}(F \star G)$.

- Narcowich (1988): Let F be a Wigner function. Under what conditions is the convolution $F \star G$ with any other Wigner function G again a Wigner function?
  - (i) $0 \in \mathrm{Spec}(F)$
  - (ii) $2\hbar \in \mathrm{Spec}(F)$
- Wigner-positive states from convolution:
  - (i) If $\{-\hbar, 0, \hbar\} \subset \mathrm{Spec}(F)$ and $\{-\hbar, \hbar\} \subset \mathrm{Spec}(G)$, then $\{-2\hbar, -\hbar, 0, \hbar, 2\hbar\} \subset \mathrm{Spec}(F \star G)$

- Wigner-positive states from convolution (cont.):
  (ii) If $\{-2\hbar, -\hbar, \hbar, 2\hbar\} \subset \text{Spec}(F)$ and $\{-\hbar, \hbar\} \subset \text{Spec}(G)$, then $\{-3\hbar, -2\hbar, -\hbar, 0, \hbar, 2\hbar, 3\hbar\} \subset \text{Spec}(F \star G)$

  Notice that the Husimi function is obtained through the procedure (i). In both cases (i) and (ii) we have automatically more elements in $\text{Spec}(F \star G)$ than we need. An alternative is:

  (iii) If $\left\{-\frac{\hbar}{2}, \frac{\hbar}{2}\right\} \subset \text{Spec}(F)$ and $\left\{-\frac{\hbar}{2}, \frac{\hbar}{2}\right\} \subset \text{Spec}(G)$, then $\{-\hbar, 0, \hbar\} \subset \text{Spec}(F \star G)$

- Question: Are there any Wigner functions F such that $\{-\frac{\hbar}{2}, \frac{\hbar}{2}\} = \text{Spec}(F)$? The answer is yes!

Theorem (N.C. Dias, JNP, 2009)

Let $\widehat{\rho} = |f><f|$ be a pure state and $\alpha \in \mathbb{R} \setminus \{0\}$. Then:

(i) *If f is a Gaussian state, then Spec$(W_\alpha f) = [-\alpha, \alpha]$*

(ii) *If f is a non-Gaussian state, then Spec$(W_\alpha f) = \{-\alpha, \alpha\}$*

(IV) (Cont.) One can also obtain Wigner-positive states through **Weyl-quantization on non-standard symplectic vector spaces.**

- Symplectic vector space $(\mathbb{R}^{2n}; \omega)$ with symplectic form:

$$\omega(z, z') = z \cdot \Omega^{-1} z' \ , \ \forall z, z' \in \mathbb{R}^{2n}$$

  $\Omega \in Gl(2n; \mathbb{R})$ is skew-symmetric.
- Symplectic group: $S \in Sp(2n; \omega)$ if

$$\omega(Sz, Sz') = \omega(z, z') \ , \ \forall z, z' \in \mathbb{R}^{2n}$$
$$\Leftrightarrow S\Omega S^T = \Omega$$

- P is ω-anti-symplectic if

$$\omega(Sz, Sz') = -\omega(z, z') \ , \ \forall z, z' \in \mathbb{R}^{2n}$$
$$\Leftrightarrow S\Omega S^T = -\Omega$$

- The standard symplectic form is:

$$\sigma(z, z') = z \cdot J^{-1} z' = p \cdot x' - x \cdot p' \ ,$$

where $z = (x, p)$, $z' = (x', p')$ and $J = -J^T = -J^{-1}$ is the standard symplectic matrix.

- Every σ-anti-symplectic matrix P can be written as

$$P = SR = RS'$$

where $S, S' \in Sp(2n; \sigma)$ and R is the matrix which inverts the momentum:

$$R = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

- Darboux theorem: There exists a linear isomorphism $\phi : (\mathbb{R}^{2n}; \omega) \to (\mathbb{R}^{2n}; \sigma)$, such that

$$\phi^* \sigma = \omega$$

Thus, there exists $D \in Gl(2n; \mathbb{R}^{2n})$ such that:

$$\Omega = DJD^T$$

We call ϕ a **Darboux map** and D a **Darboux matrix.** The Darboux matrix is not unique: D, D' are Darboux matrices if and only if $(D')^{-1}D$, $D^{-1}D' \in Sp(2n; \sigma)$. The set of all Darboux matrices is denoted by $\mathcal{D}(2n; \omega)$.

(A) Weyl quantization on $(\mathbb{R}^{2n};\sigma)$

- Heisenberg algebra: Let $(\widehat{I}\psi)(x)=\psi(x)$, $(\widehat{q}_j\psi)(x)=x_j\psi(x)$ and $(\widehat{p}_k\psi)(x)=-i\partial_k\psi(x)$, $j,k=1,\cdots,n$. Set $\widehat{Z}=(\widehat{q},\widehat{p})$.

$$\left[\widehat{Z}_\alpha,\widehat{Z}_\beta\right]=iJ_{\alpha\beta}\widehat{I}\,,\ \left[\widehat{Z}_\alpha,\widehat{I}\right]=0\,,\ \forall\alpha,\beta=1,\cdots,2n$$

- Heisenberg-Weyl displacement operators: for $\xi=(x_0,p_0)\in\mathbb{R}^{2n}$

$$(\widehat{D}^\sigma(\xi)\psi)(x)=(e^{i\sigma(\xi,\widehat{Z})}\psi)(x)=$$

$$=(e^{i(p_0\cdot\widehat{q}-x_0\cdot\widehat{p})}\psi)(x)=e^{ip_0\cdot x-\frac{i}{2}p_0\cdot x_0}\psi(x-x_0)$$

- Commutation relations:

$$\widehat{D}^\sigma(\xi)\widehat{D}^\sigma(\zeta)=e^{\frac{i}{2}\sigma(\xi,\zeta)}\widehat{D}^\sigma(\xi+\zeta)=e^{i\sigma(\xi,\zeta)}\widehat{D}^\sigma(\zeta)\widehat{D}^\sigma(\xi)$$

- Schrödinger representation of the Heisenberg group $\mathbb{H}(n)$:

$$\widehat{V}^{\sigma}(\xi,\tau) = e^{i\tau}\widehat{D}^{\sigma}(\xi) \ , \ (\xi,\tau) \in \mathbb{R}^{2n+1}$$

$$(\xi,\tau)\cdot(\xi',\tau') = \left(\xi+\xi',\tau+\tau'+\frac{1}{2}\sigma(\xi,\xi')\right)$$

- Weyl operators: Given $a^{\sigma} \in \mathcal{S}'(\mathbb{R}^{2n})$:

$$\widehat{A} := \left(\frac{1}{2\pi}\right)^{n}\int_{\mathbb{R}^{2n}} \mathcal{F}_{\sigma}a^{\sigma}(z)\widehat{D}^{\sigma}(z)dz$$

- Weyl correspondence: $\mathcal{S}'(\mathbb{R}^{2n}) \to \mathcal{L}\left(\mathcal{S}(\mathbb{R}^{n}),\mathcal{S}'(\mathbb{R}^{n})\right)$

$$a^{\sigma} \overset{\text{Weyl}}{\longleftrightarrow} \widehat{A} \quad \text{or} \quad \widehat{A} \overset{\text{Weyl}}{\longleftrightarrow} a^{\sigma}$$

- Wigner functions:

$$\widehat{\rho}_\psi = |\psi >< \psi| \quad \stackrel{\text{Weyl}}{\longleftrightarrow} (2\pi)^n W_\sigma \psi(z)$$

$$\widehat{\rho} = \sum_j p_j |\psi_j >< \psi_j| \quad \stackrel{\text{Weyl}}{\longleftrightarrow} (2\pi)^n W_\sigma \rho(z) = (2\pi)^n \sum_j p_j W_\sigma \psi_j(z)$$

- Expectation values:

$$< \widehat{A} >_\rho = Tr(\widehat{A}\widehat{\rho}) = \int_{\mathbb{R}^{2n}} a^\sigma(z) W_\sigma \rho(z) dz$$

(B) Weyl quantization on $(\mathbb{R}^{2n};\omega)$

- Modified Heisenberg algebra: Let $\widehat{\Xi} = P\widehat{Z}$, where $P \in \mathcal{D}(2n;\omega)$.

$$\left[\widehat{\Xi}_\alpha, \widehat{\Xi}_\beta\right] = i\Omega_{\alpha\beta}\widehat{I}\ , \ \left[\widehat{\Xi}_\alpha, \widehat{I}\right] = 0\ , \ \forall \alpha, \beta = 1, \cdots, 2n$$

- Modified Heisenberg-Weyl displacement operators: for $\xi = (x_0, p_0) \in \mathbb{R}^{2n}$:

$$\widehat{D}^\omega(\xi) = e^{i\omega(\xi,\widehat{\Xi})} = e^{i\sigma(P^{-1}\xi,\widehat{Z})} = \widehat{D}^\sigma(P^{-1}\xi)$$

- Commutation relations:

$$\widehat{D}^\omega(\xi)\widehat{D}^\omega(\zeta) = e^{\frac{i}{2}\omega(\xi,\zeta)}\widehat{D}^\omega(\xi+\zeta) = e^{i\omega(\xi,\zeta)}\widehat{D}^\omega(\zeta)\widehat{D}^\omega(\xi)$$

- Schrödinger representation of the modified Heisenberg group $\mathbb{H}^{\omega}(n)$:

$$\widehat{V}^{\omega}(\xi,\tau) = e^{i\tau}\widehat{D}^{\omega}(\xi)\ ,\ (\xi,\tau)\in\mathbb{R}^{2n+1}$$

$$(\xi,\tau)\cdot(\xi',\tau') = \left(\xi+\xi',\tau+\tau'+\frac{1}{2}\omega(\xi,\xi')\right)$$

- Weyl operators: Given $a^{\sigma}\in\mathcal{S}'(\mathbb{R}^{2n})$:

$$\widehat{A} := \left(\tfrac{1}{2\pi}\right)^{n}\int_{\mathbb{R}^{2n}}\mathcal{F}_{\sigma}a^{\sigma}(z)\widehat{D}^{\sigma}(z)dz =$$

$$= \frac{1}{(2\pi)^{n}\sqrt{\det\Omega}}\int_{\mathbb{R}^{2n}}\mathcal{F}_{\sigma}a^{\sigma}(P^{-1}z')\widehat{D}^{\omega}(z')dz'$$

On the other hand:

$$\mathcal{F}_{\sigma}a^{\sigma}(P^{-1}z) = \frac{1}{(2\pi)^{n}}\int_{\mathbb{R}^{2n}}e^{-i\sigma(P^{-1}z,z')}a^{\sigma}(z')dz' =$$

$$= \frac{1}{(2\pi)^{n}\sqrt{\det\Omega}}\int_{\mathbb{R}^{2n}}e^{-i\omega(z,u)}a^{\sigma}(P^{-1}u)du$$

This suggests the following definition of ω-symplectic Fourier transform and ω-Weyl Symbol:

$$\mathcal{F}_\omega a(z) := \frac{1}{(2\pi)^n \sqrt{\det \Omega}} \int_{\mathbb{R}^{2n}} e^{-i\omega(z,z')} a(z') dz'$$

$$a^\omega(z) = a^\sigma(P^{-1}z)$$

Thus:

$$\widehat{A} = \frac{1}{(2\pi)^n \sqrt{\det \Omega}} \int_{\mathbb{R}^{2n}} \mathcal{F}_\omega a^\omega(z) \widehat{D}^\omega(z) dz$$

- Expectation values:

$$< \widehat{A} >_\rho = \int_{\mathbb{R}^{2n}} a^\sigma(z) W_\sigma \rho(z) dz =$$

$$= \frac{1}{\sqrt{\det \Omega}} \int_{\mathbb{R}^{2n}} a^\sigma(P^{-1}z') W_\sigma \rho(P^{-1}z') dz' =$$

$$= \int_{\mathbb{R}^{2n}} a^\omega(z) W_\omega \rho(z) dz$$

where

- Wigner functions:

$$W_\omega \rho(z) := \frac{1}{\sqrt{\det \Omega}} W_\sigma \rho(P^{-1}z)$$

- Darboux Invariance: Expectation values, probabilities are independent of $P \in \mathcal{D}(2n;\omega)$.

- Examples:

Example 1: 2 subsystems - Alice with n_A degrees of freedom and Bob with n_B degrees of freedom; $n = n_A + n_B$.

$$z_A = (x_A, p_A) \in \mathbb{R}^{2n_A} \ , \ z_B = (x_B, p_B) \in \mathbb{R}^{2n_B}$$

$$z = (x_A, x_B, p_A, p_B) \in \mathbb{R}^{2n}$$

Partial transpose (reversal of Bob's momentum):

$$z \mapsto Pz = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -I \end{pmatrix} \begin{pmatrix} x_A \\ x_B \\ p_A \\ p_B \end{pmatrix} = \begin{pmatrix} x_A \\ x_B \\ p_A \\ -p_B \end{pmatrix}$$

Note that P is neither σ-symplectic nor σ-anti-symplectic. Thus $W_\sigma\rho(P^{-1}z)$ is not, in general, a σ-Wigner function.
Rather, we can think of P as Darboux matrix for the symplectic form:

$$\omega(z,z') = z \cdot \Omega^{-1} z' \ ,$$

with

$$\Omega = PJP^T = \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & -I \\ -I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix}$$

And:

$$W_\sigma\rho(P^{-1}z) = W_\omega\rho(z)$$

States which satisfy the PPT criterion (and in particular, separable states) lie in the intersection of the sets $\mathcal{W}_\sigma(\mathbb{R}^{2n})$ of σ-Wigner functions and $\mathcal{W}_\omega(\mathbb{R}^{2n})$ of ω-Wigner functions.

- Examples (cont.)

Example 2: Let

$$S^{(1)} = \begin{pmatrix} A^{(1)} & B^{(1)} \\ C^{(1)} & D^{(1)} \end{pmatrix} , \quad \text{and} \quad S^{(2)} = \begin{pmatrix} A^{(2)} & B^{(2)} \\ C^{(2)} & D^{(2)} \end{pmatrix}$$

be two free symplectic matrices ($\det B^{(1)} \neq 0$, $\det B^{(2)} \neq 0$)
Let $\widehat{S^{(1)}}$, $\widehat{S^{(2)}}$ be associated metaplectic operators and set

$$\widehat{\psi}_{S^{(1)}}(\xi) := \left(\widehat{S^{(1)}}\psi\right)(\xi) \, , \; \widehat{\psi}_{S^{(2)}}(\eta) := \left(\widehat{S^{(2)}}\psi\right)(\eta)$$

$$D^{(1,2)} := \begin{pmatrix} A^{(1)} & B^{(1)} \\ A^{(2)} & B^{(2)} \end{pmatrix}$$

Assume that $\det D^{(1,2)} \neq 0$ and that it is neither σ-symplectic nor σ-anti-symplectic. Again, we define:

$$\omega(z,z') = z \cdot \Omega^{-1} z' \ , \ \Omega = D^{(1,2)} J \left(D^{(1,2)} \right)^T$$

Then, the associated Wigner function $W_\omega \psi(z) = \frac{1}{\sqrt{\det \Omega}} W_\sigma \psi \left(D^{(1,2)} z \right)$ has the following marginals:

$$\int_{\mathbb{R}^n} W_\omega \psi(\xi,\eta) d\eta = |\widehat{\psi}_{S^{(1)}}(\xi)|^2$$

$$\int_{\mathbb{R}^n} W_\omega \psi(\xi,\eta) d\xi = |\widehat{\psi}_{S^{(2)}}(\eta)|^2$$

This generalizes:

$$\int_{\mathbb{R}^n} W_\sigma \psi(x,p)dp = |\psi(x)|^2$$

$$\int_{\mathbb{R}^n} W_\sigma \psi(x,p)dx = |\mathcal{F}_\hbar \psi(p)|^2$$

- Examples (cont.)

Example 3: A modified Heisenberg-algebra (noncommutative QM, string theory, quantum Hall effect, etc.)

$$\left[\widehat{q}_1,\widehat{q}_2\right]=i\theta\widehat{\mathrm{I}}\ ,\ \left[\widehat{q}_1,\widehat{p}_1\right]=\left[\widehat{q}_2,\widehat{p}_2\right]=i\hbar\widehat{\mathrm{I}}\ ,$$

$$\left[\widehat{p}_1,\widehat{p}_2\right]=\left[\widehat{q}_1,\widehat{p}_2\right]=\left[\widehat{q}_2,\widehat{p}_1\right]=0$$

Symplectic form:

$$\Omega=\begin{pmatrix} 0 & \theta & 0 & \hbar \\ -\theta & 0 & \hbar & 0 \\ 0 & -\hbar & 0 & 0 \\ -\hbar & 0 & 0 & 0 \end{pmatrix}$$

It can be shown that:

$$W_\omega\psi(x,p)=\frac{1}{(2\pi\hbar)}\int_{\mathbb{R}^{\not=}}\psi(x+y/2)\star_\theta\overline{\psi(x-y/2)}e^{-\frac{i}{\hbar}y\cdot p}dy$$

where

$$\psi(x+y/2)\star_\theta \overline{\psi(x-y/2)} = \psi(x+y/2)\overline{\psi(x-y/2)} +$$

$$\frac{i\theta}{2}\left(\frac{\partial\psi}{\partial x_1}(x+y/2)\overline{\frac{\partial\psi}{\partial x_2}(x-y/2)} - \frac{\partial\psi}{\partial x_2}(x+y/2)\overline{\frac{\partial\psi}{\partial x_1}(x-y/2)}\right) + \mathcal{O}(\theta^2)$$

We have in particular:

$$\int_{\mathbb{R}^2} W_\omega\psi(x,p)dp = \psi(x)\star_\theta \overline{\psi(x)}$$

Thus, in general, the marginal distribution is not everywhere non-negative, but it is of the θ-positive type.

- Conclusion: If $W_\omega\psi$ is simultaneously a ω-Wigner function and a σ-Wigner function, then its marginal distribution is a positive σ-Wigner function for a particle with position $x = x_1$ and momentum $p = x_2$, provided we set $\theta = \hbar$.

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