enjoy
the quantum
Quantumness, quantum non-Gaussianity and Wigner negativity

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your pic may be here, we are hiring!
This talk is about tension between the notion of non-classicality arising from phase-space description of CV systems and those coming from information-theoretic considerations.
Classical states: those admitting a description in terms of a mixture of coherent states

\[ \rho = \int d^2 \alpha \, P(\alpha) \, |\alpha\rangle \langle \alpha| \]

\[ P(\alpha) > 0 \quad 1 = \int d^2 \alpha \, P(\alpha) \]

Physics based notion (phenomenology of classical states may be described by ME, no need of quantum "optics").

Nonclassical states: those with a negative/singular Glauber P-function (oops, Glauber-Sudarshan)
Yet, a plethora of (inequivalent) notions of nonclassicality exist.

Negativity of Wigner function  
(squeezed states are classical?)

Violation of Bell inequalities  
(no violation with Gaussian states & measurements)

Separability

Vanishing quantum discord  
(\( |n\rangle \langle n| \otimes |n\rangle \langle n| \) is a classical state?)

Quantum non-Gaussianity & stellar rank  
(quantum sampling, which task is more nonclassical?)

Quadrature coherence scale  
cannot find a fancy pic for this
Non-classicality criteria from phase-space representations and information-theoretical constraints are maximally inequivalent


Resource theories of quantum non-Gaussianity and Wigner negativity

Non-classicality criteria from phase-space representations and information-theoretical constraints are maximally inequivalent


Quantumness of bipartite systems made of two bosonic modes

\[ \rho_{AB} \]

\[ \Omega_{AB} \]

\[ [a, a^\dagger] = 1 \]

\[ [b, b^\dagger] = 1 \]
P-classical states

Convex combination of factorized coherent states

\[ \rho_{AB} = \int \int d^2\alpha d^2\beta P(\alpha, \beta) |\alpha\rangle \langle \alpha| \otimes |\beta\rangle \langle \beta| \]

\[ P(\alpha, \beta) > 0 \quad 1 = \int \int d^2\alpha d^2\beta P(\alpha, \beta) \]
Entanglement & Discord

(which notion is captured?)

Separable states (zero entanglement): can be created from any other state by local operations and classical communication

$$Q_{AB} = \sum_k p(k) Q_{Ak} \otimes Q_{Bk}$$

Classical-Quantum states (zero A-discord): there exists a basis for A for which the locally-accessible information is maximal and can be obtained without disturbance to the combined system

$$Q_{AB} = \sum_a p_A(a) |\varphi_a\rangle \langle \varphi_a| \otimes Q_{Ba}$$

Quantum-classical states (zero B-discord): there exists a basis for B for which the locally-accessible information is maximal and can be obtained without disturbance to the combined system

$$Q_{AB} = \sum_b p_B(b) Q_{Ab} \otimes |\psi_b\rangle \langle \psi_b|$$
Quantum Discord

- (classical) Mutual information $I_{AB}$

$$I_{AB} = H_A + H_B - H_{AB}$$

$$= H_B - H_{B|A}$$

$$= H_A - H_{A|B}$$

$$p_{AB}(a, b) = p_{A|B}(a|b) p_B(b)$$

$$= p_{B|A}(b|a) p_A(a)$$

Information gain about a subsystem as a results of a measurement on the other
Quantum Discord

(quantum) Mutual information $\mathcal{Q}_{AB}$

$$I_{AB} = S_A + S_B - S_{AB}$$
$$\neq S_B - S_{B|A}$$
$$\neq S_A - S_{A|B}$$

$$p_A(a) = \text{Tr}_{AB} [\mathcal{Q}_{AB} \Pi_a \otimes \mathbb{I}]$$

$$\mathcal{Q}_{Ba} = \frac{1}{p_A(a)} \text{Tr}_A [\mathcal{Q}_{AB} \Pi_a \otimes \mathbb{I}]$$
Quantum Discord

- $S_B - S_{B|A}, S_A - S_{A|B}$ depend on the choice of measurement

- The symmetry is broken $S_B - S_{B|A} \neq S_A - S_{A|B}$

- There are different from $I_{AB}$

\[
D_{B|A} = I_{AB} - \sup_{\{\Pi_a\}} (S_B - S_{B|A}) = S_A - S_{AB} + \inf_{\{\Pi_a\}} \sum_a p_A(a) S(q_{Ba}) > 0
\]

Quantum A-discord

\[
D_{A|B} = I_{AB} - \sup_{\{\Pi_b\}} (S_A - S_{A|B}) = S_B - S_{AB} + \inf_{\{\Pi_b\}} \sum_b p_B(b) S(q_{Ab}) > 0
\]

Quantum B-discord
Entanglement & Discord
(which notion is captured?)

- Separable states (zero entanglement): can be created from any other state by local operations and classical communication

\[ \rho_{AB} = \sum_{k} p(k) \rho_{Ak} \otimes \rho_{Bk} \]

- Classical-Quantum states (zero A-discord): there exists a basis for A for which the locally-accessible information is maximal and can be obtained without disturbance to the combined system

\[ \rho_{AB} = \sum_{a} p_{A}(a) |\varphi_{a}\rangle\langle \varphi_{a}| \otimes \rho_{Ba} \]

- Quantum-classical states (zero B-discord): there exists a basis for B for which the locally-accessible information is maximal and can be obtained without disturbance to the combined system

\[ \rho_{AB} = \sum_{b} p_{B}(b) \rho_{Ab} \otimes |\psi_{b}\rangle\langle \psi_{b}| \]
Classical-Classical (CC-) states

Classical-classical states (zero discord): correlations are completely specified by the knowledge of a classical (density) distribution

$$\rho_{AB} = \sum_{ab} p_{AB}(a, b) |\phi_a\rangle \langle \phi_a| \otimes |\psi_b\rangle \langle \psi_b|$$

$$\rho_{AB} = \int \int da \; db \; F(a, b) |a\rangle \langle a| \otimes |b\rangle \langle b|$$
Classical-Classical states
(examples of two-mode CC states)

\[ \rho_{AB} = \sum_n p_n |n\rangle\langle n| \otimes |n\rangle\langle n| \] number correlated states

\[ \rho_{AB} = \sum_{nm} p_{nm} |n\rangle\langle n| \otimes |m\rangle\langle m| \]
P-classical vs CC-classical

Convex combination of factorized coherent states

$$\rho_{AB} = \int \int d^2 \alpha \, d^2 \beta \, P(\alpha, \beta) \, |\alpha\rangle \langle \alpha| \otimes |\beta\rangle \langle \beta|$$

Convex combination of factorized orthogonal states

$$\rho_{AB} = \int \int da \, db \, F(a, b) \, |a\rangle \langle a| \otimes |b\rangle \langle b|$$
Difference photocurrent

For P-classical states

\[ O_D = a^\dagger a - b^\dagger b \]

\[ \Delta O_D^2 = |\alpha_0|^2 + |\beta_0|^2 + \text{Tr } C \geq |\alpha_0|^2 + |\beta_0|^2 > 0 \]

Each mode has a fluctuating number of quanta and the difference should fluctuate accordingly: for a classical two-mode system the amount of intensity correlations is bounded.

For number correlated states (CC states)

\[ q_{AB} = \sum_n p_n |n\rangle \langle n| \otimes |n\rangle \langle n| \]

\[ \Delta O_D^2 = 0 \]
Nonclassical conditional states

- Only states violating P criterion may lead to the conditional generation of genuine quantum states with no classical analogue.

- For number correlated states (CC states) \( Q_{AB} = \sum_n p_n |n\rangle\langle n| \otimes |n\rangle\langle n| \)

Measuring the photon number on one side produces nonclassical (Fock) states.
P-classical states are not CC

\[ \varrho_{AB} = \sum_b p_{AB}(a, b) |\phi_a\rangle \langle \phi_a| \otimes |\psi_b\rangle \langle \psi_b| \]

Necessary condition for CC: All possible conditioned states of B mutually commute. This can be seen by applying any POVM on A: any state of B conditioned on any outcome at A remains diagonal in the original basis.

\[ \varrho_{AB} = \int \int d^2 \alpha d^2 \beta P(\alpha, \beta) |\alpha\rangle \langle \alpha| \otimes |\beta\rangle \langle \beta| \]

\[ \varrho_A = \text{Tr}_A [\varrho_{AB}] \quad \varrho_0 = \text{Tr}_A [\varrho_{AB} |0\rangle \langle 0|] \]

\[ [\varrho_A, \varrho_0] = 0 \quad \text{This is a condition not satisfied by a generic well behaved P-function: almost all P-classical states are not CC} \]

\[ \int d^2 \alpha d^2 \alpha' d^2 \beta d^2 \beta' P(\alpha, \beta) P(\alpha', \beta') e^{-|\alpha|^2} e^{-|\alpha'|^2} e^{-|\beta|^2} e^{-|\beta'|^2} (e^{\alpha \overline{\alpha'}} - \text{c.c.}) = 0. \]
CC states are not P-classical

- The set of single mode P-classical states is nowhere dense in the bosonic space (its closure has no interior points).

- Partial traces of P-classical states should be single-mode P-classical states.

- By construction, one shows that P-classical states are nowhere dense in the set of CC states i.e. generic CC states are not P-classical.
P- vs CC-classicality

- CC criterion looks at the correlations between the information of A and B, as encoded in their states and regardless the quantumness of the states themselves.

- P criterion takes into account physical constraints on those as well: e.g. creating Fock states with the same number of quanta does correspond to establishing quantum correlations between the modes, irrespectively from the fact that the information needed to perform this action may be of purely classical (local) origin.

- The set of states being simultaneously P-classical and CC is negligible (either in a metrical or topological sense).

- P-classical states violate CC criterion: they represent an experimentally cheap resource in communication protocols that require security against local broadcasting.
From a fundamental physical point of view, discord (and more in general any information-theoretical quantity) appears unable to account for the very physical constraints involved in the establishment of correlations.

Allegedly classical correlations established between systems prepared in states with no classical analogue are quantum in nature.
P- vs CC-classicality

“Information is physical”

(R. Landauer 1991)

“... and physics is not merely information.”

(me and others, over the years)
Resource theories of quantum non-Gaussianity and Wigner negativity


Some interesting states

- A positive Wigner function is a sufficient condition to have a quantum system that can be efficiently simulated by classical algorithms.

- For pure states, WignerP is equivalent to Gaussianity, but in general not all WignerP states can be generated using GPs.

  - Gaussian unitaries (e.g., displacement, squeezing, CZ, ...)
  - Composition with pure Gaussian states (e.g., squeezed states)
  - Pure Gaussian measurements on subsystems (e.g., homodyne)
  - Partial trace on subsystems

- One is led to consider the convex hull of GSs, as the maximal set of states that can be generated by GPs.
Intuitive intermezzo

QnG states with positive Wigner function are bounded resources in the same sense in which entangled with positive partial transpose are bounded resources.
Some interesting states

Convex hull of Gaussian states

\[ \mathcal{G} = \left\{ \rho \in S(\mathcal{H}) \mid \rho = \int d\lambda \, p(\lambda) \, |\psi_G(\lambda)\rangle \langle \psi_G(\lambda)| \right\} \]

Wigner-positive states

\[ \mathcal{W}_+ = \{ \rho \in S(\mathcal{H}) \mid W_\rho(r) \geq 0 \} \]
Resource Theory

State space

Free operations

Free states

Resources are quantified by "resource monotone"

(1) $M(\rho) = 0$ for free states

(2) Monotonicity under deterministic Gaussian protocols

(3) Monotonicity on average under probabilistic Gaussian protocols
A W-negativity monotone

(Wigner Log negativity aka CV-mana)

\[ M(\rho) = \log \left( \int dr \ |W_\rho(r)| \right) \]

- inspired by negative volume of Wigner function (Kenfack 04)
- not convex as it happens for the logarithmic negativity of entanglement (Plenio 05)
- example: cubic phase state

\[ |\gamma, r\rangle = \exp[i\gamma \hat{x}^3] \hat{S}(r)|0\rangle \]

\[ \mathcal{M}(|\gamma, r\rangle) = \mathcal{M}(|e^{3r}\gamma, 0\rangle) = f(e^{3r}\gamma) \]

\[ \hat{S}(r)^\dagger \exp[i\gamma \hat{x}^3] \hat{S}(r) = \exp[i\gamma e^{3r} \hat{x}^3] \]
A QnG monotone

Convex roof of relative entropy of nG

\[ \delta_{\text{CR}}[\rho] = \inf_{p_i,|\psi_i\rangle} \sum_i p_i \delta[|\psi_i\rangle] \]

\[ \delta[|\psi\rangle] = S(\rho||\tau_G) = S(\tau_G) \]

(\text{Genoni & Paris 2008})

- where \( S(\rho||\sigma) = \text{Tr}[\rho(\log \rho - \log \sigma)] \) is the quantum relative entropy, \( S(\rho) = \text{Tr}(\rho \log \rho) \) is the von Neumann entropy, and \( \tau_G \) is the reference (mixed) Gaussian state having the same covariance matrix as \( |\psi\rangle \).

- example: cubic phase state

\[ |\gamma, r\rangle = \exp[i \gamma \hat{x}^3] \hat{S}(r) |0\rangle \]

\[ \delta[|\gamma, r\rangle] = \hbar(\sqrt{1 + 9(e^{3r} \gamma)^2}) \]
As a consequence the contour lines of any monotone on the quantum non-Gaussianity to the resource theory of coherence is replaced by an integral. The functional construction and property of the squeezing operator $\hat{S}_r$ is given by:

$$\hat{S}_r = \hat{S}_r \hat{\gamma} \hat{\gamma}^\dagger.$$ 

where the squeezing operator $\hat{S}_r$ is defined as

$$\hat{S}_r = \hat{S}_r \hat{\gamma} \hat{\gamma}^\dagger,$$

and $\hat{\gamma}$ is the annihilation operator.

In particular, besides the aforementioned class of cubic-phase gate, we now use the WLN to assess the resourcefulness of some paradigmatic examples of non-Gaussian states. In fact, we can easily compute the non-Gaussianity ($W$) and the nonlinearity ($\delta$) for the following states:

- Photon-added states
- Photon-subtracted states
- Cubic-phase states

These states are relevant in the general framework just introduced, and we expect that they might have different qualitative behavior. We remark that the same fact has also been observed for ground states of anharmonic potentials.

In Fig. 3, we illustrate the non-Gaussianity ($W$) and the nonlinearity ($\delta$) for different values of the parameters as a function of $ye^3r$. The two parametric curves $W$ and $\delta$ have the same qualitative behavior.

Moreover, this is a function of a single parameter in both cases. In Fig. 3(a), we use the WLN for $W$ and $\delta$, and the non-Gaussianity ($W$) is formally mean that we can freely interconvert the two figures of merit $W$ and $\delta$.

In Fig. 3(b), we observe that $W$ and $\delta$ can easily be computed for ground states of anharmonic potentials. In particular, we can compute the non-Gaussianity ($W$) for the following states:

- Photon-added states
- Photon-subtracted states
- Cubic-phase states

These states are relevant in the general framework just introduced, and we expect that they might have different qualitative behavior. We remark that the same fact has also been observed for ground states of anharmonic potentials.

In Fig. 3(c), we illustrate the non-Gaussianity ($W$) and the nonlinearity ($\delta$) for different values of the parameters as a function of $ye^3r$. The two parametric curves $W$ and $\delta$ have the same qualitative behavior.

Moreover, this is a function of a single parameter in both cases. In Fig. 3(d), we use the WLN for $W$ and $\delta$, and the non-Gaussianity ($W$) is formally mean that we can freely interconvert the two figures of merit $W$ and $\delta$.
Random remarks

- There are no "maximal resourcefulness" states.


- Up to some requests WLN is the unique "decent" W-p monotone.

- WLN is useful to assess negativity concentration protocols.
Conclusions?

Physics is not merely information: quantumness of correlations cannot be the unique ingredients in detecting nonclassical behaviour.

Phase-space is not enough: when interested in features related to information content, resource theory is a crucial tool.
thank you