Geometry of the Kirkwood-Dirac-positive states

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arXiv:2306.00086

QUIDIQUA, Lille, France
10 November 2023

The Kirkwood-Dirac (KD) distribution: definition

- $\mathcal{H}$: Hilbert space of finite dimension d. $(|a_i\rangle)_{i\in[\![1,d]\!]}$ and $(|b_j\rangle)_{j\in[\![1,d]\!]}$ two orthonormal bases of $\mathcal{H}$. $U_{ij} = (\langle a_i|b_j\rangle)$ transition matrix.
- Examples :
  - → $d = 2$, $(|0\rangle, |1\rangle)$ and $(|+\rangle, |-\rangle)$;
  - → $d \geqslant 2$, think of U as Discrete Fourier Transform (DFT).
- The Kirkwood(1933)-Dirac(1945) quasiprobability distribution of ρ :

$$\forall (i,j) \in [\![1,d]\!]^2,\ Q_{ij}(\rho) = \langle b_j|a_i\rangle\langle a_i|\rho|b_j\rangle \in \mathbb{C}.$$

$$\sum_{i=1}^{d} Q_{ij}(\rho) = \langle b_j|\rho|b_j\rangle,\ \sum_{j=1}^{d} Q_{ij}(\rho) = \langle a_i|\rho|a_i\rangle \text{ and } \sum_{i,j} Q_{ij}(\rho) = 1.$$

- Analogous to the Wigner function. However, more flexible as not designed only for position and momentum.

KD positivity

- ρ is a KD-positive state if $Q_{ij}(\rho) \geqslant 0$ for all $(i,j) \in [\![1,d]\!]^2$.

Examples : $Q_{kl}(|a_1\rangle\langle a_1|) = \delta_{1,k} \left|\langle a_1|b_l\rangle\right|^2$. $|a_1\rangle\langle a_1|$ KD-positive.

$\mathcal{A} = \{|a_i\rangle\langle a_i| \mid i \in [\![1,d]\!]\}$ and $\mathcal{B} = \{|b_j\rangle\langle b_j| \mid j \in [\![1,d]\!]\}$ KD-positive.

$\mathcal{E}_{\mathrm{KD}+}$ is the set of KD-positive states.

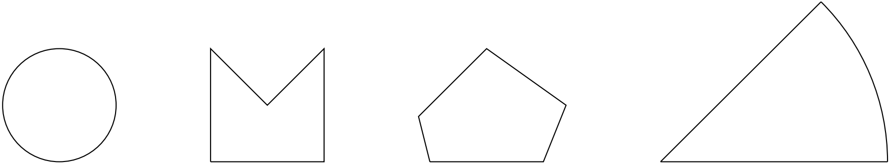
- $\mathcal{E}_{\mathrm{KD}+}$ is convex : if $\rho_1, \rho_2 \in \mathcal{E}_{\mathrm{KD}+}$, for $\lambda \in [0,1]$, $\lambda\rho_1 + (1-\lambda)\rho_2 \in \mathcal{E}_{\mathrm{KD}+}$.

GOAL: obtain a precise/complete description of the geometry of $\mathcal{E}_{\mathrm{KD}+}$.

- KD-nonpositive states linked with quantum advantages

Geometry of convex sets (1)

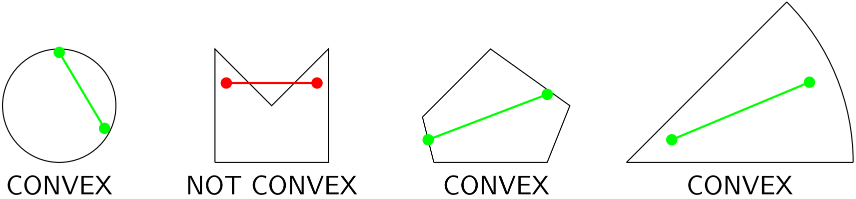
- A convex set: the segment between two points in the set lies in the set.



CONVEX OR NOT CONVEX ?

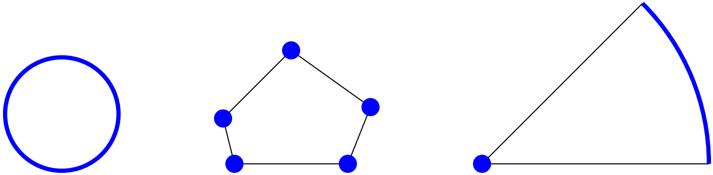
Geometry of convex sets (2)

- A convex set: the segment between two points in the set lies in the set.



Geometry of convex sets (3)

- A convex set: The segment between two points in the set lies in the set.



- ρ is extreme: if $\rho = \lambda\rho_1 + (1-\lambda)\rho_2$ then $\rho_1 = \rho$ and $\rho_2 = \rho$.

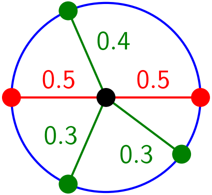
Theorem (Krein-Milman)

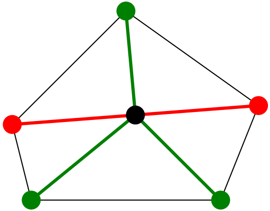
$$K \textit{ is convex, closed and bounded} \Rightarrow K = \mathrm{conv}\left(K^{\mathrm{ext}}\right)$$

where K^{ext} *is the set of extreme points of* K *and* $\mathrm{conv}\left(\right)$ *is the convex hull.*

Geometry of convex sets (4)

- A convex set: The segment between two points in the set lies in the set.





- $\mathrm{conv}\left(K_{ext}\right) = \bigcup_{n \in \mathbb{N}^*} \left\{ \sum_{p=1}^{n} \lambda_p \rho_p, \rho_p \in K_{ext}, \sum_{p=1}^{n} \lambda_p = 1, \lambda_p \geqslant 0 \right\}$

- **The decomposition may not be unique !**

Looking for the extreme points of $\mathcal{E}_{\mathrm{KD}+}$ (1)

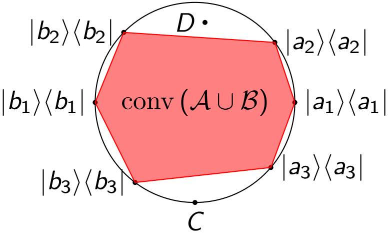
- KD distribution: $Q_{ij}(\rho) = \langle b_j | a_i \rangle \langle a_i | \rho | b_j \rangle$.
- The extreme points of the set of quantum states = the pure states.
- **FACT:** Pure KD positive states are extreme states of $\mathcal{E}_{\mathrm{KD}+}$.
- Denote by $\mathcal{E}_{\mathrm{KD}+}^{\mathrm{pure}}$ the set of pure KD positive states.
- $\mathcal{A} = \{|a_i\rangle\langle a_i| \mid i \in [\![1, d]\!]\} \subset \mathcal{E}_{\mathrm{KD}+}^{\mathrm{pure}}$; $\mathcal{B} = \{|b_j\rangle\langle b_j| \mid j \in [\![1, d]\!]\} \subset \mathcal{E}_{\mathrm{KD}+}^{\mathrm{pure}}$.

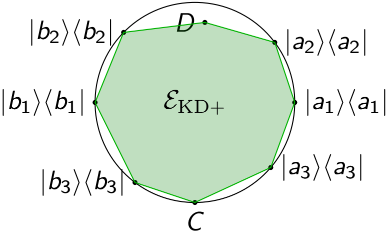
$$\mathcal{A} \cup \mathcal{B} \subsetneq \mathcal{E}_{\mathrm{KD}+}^{\mathrm{pure}} \subsetneq \mathcal{E}_{\mathrm{KD}+}^{\mathrm{ext}}.$$

- Difficulty: the space of quantum states has dimension d^2 where $d = \dim(\mathcal{H})$.

Looking for the extreme points of $\mathcal{E}_{\mathrm{KD}+}$ (2)

$$\mathrm{conv}\left(\mathcal{A}\cup\mathcal{B}\right) \subsetneq \mathrm{conv}\left(\mathcal{E}_{\mathrm{KD}+}^{\mathrm{pure}}\right) \subsetneq \mathcal{E}_{\mathrm{KD}+} = \mathrm{conv}\left(\mathcal{E}_{\mathrm{KD}+}^{\mathrm{ext}}\right).$$

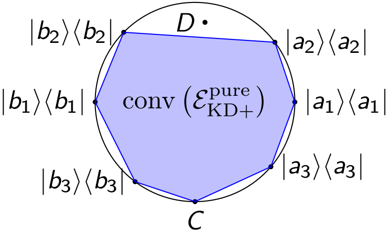


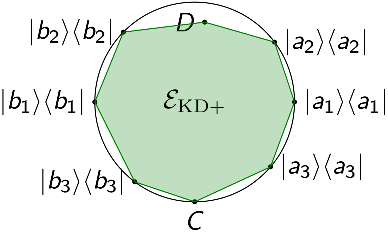


- Difficulty 1: Find or rule out the existence of pure KD-positive states that are not basis states (points like C).
- Difficulty 2: Find or rule out the existence of mixed KD-positive states (points like D).

Looking for extreme points of $\mathcal{E}_{\mathrm{KD}+}$ (3)

$$\mathrm{conv}\,(\mathcal{A} \cup \mathcal{B}) \subsetneq \mathrm{conv}\,\left(\mathcal{E}_{\mathrm{KD}+}^{\mathrm{pure}}\right) \subsetneq \mathcal{E}_{\mathrm{KD}+} = \mathrm{conv}\,\left(\mathcal{E}_{\mathrm{KD}+}^{\mathrm{ext}}\right).$$

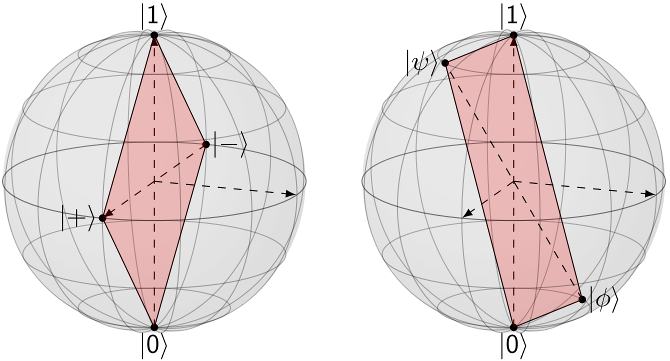




- Difficulty 1: Find or rule out the existence of pure KD-positive states that are not basis states (points like C).
- Difficulty 2: Find or rule out the existence of mixed extreme KD-positive states (points like D).

Main results: the case $d = 2$ (qubits)

If $U_{ij} = \langle a_i | b_j \rangle \neq 0$ for all $(i, j) \in [\![1, d]\!]$

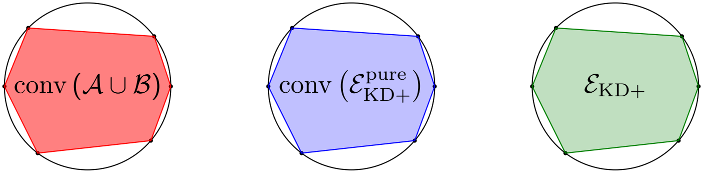


$\mathcal{A} = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$
$\mathcal{B} = \{|+\rangle\langle +|, |-\rangle\langle -|\}$

$\mathcal{A} = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$
$\mathcal{B} = \{|\phi\rangle\langle \phi|, |\psi\rangle\langle \psi|\}$

$$\mathrm{conv}\,(\mathcal{A} \cup \mathcal{B}) = \mathcal{E}_{\mathrm{KD}+}.$$

Main results: the case $d = 3$, random bases



Theorem (C.L, D.R.M. Arvidsson-Shukur, S. De Bièvre, arXiv June 2023)

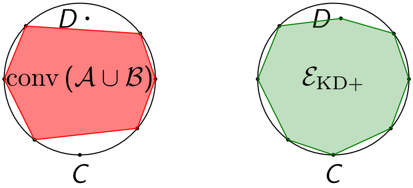
The equality

$$\mathcal{E}_{\mathrm{KD+}} = \mathrm{conv}\,(\mathcal{A} \cup \mathcal{B})$$

holds for an open dense set of unitary matrices in dimension 3.

- Take two random bases $\mathcal{A}$ and $\mathcal{B}$ (choose a random unitary matrix);
- With probability 1, one then has $\mathcal{E}_{\mathrm{KD+}} = \mathrm{conv}\,(\mathcal{A} \cup \mathcal{B})$.

Main results: the case $d = 3$, spin 1

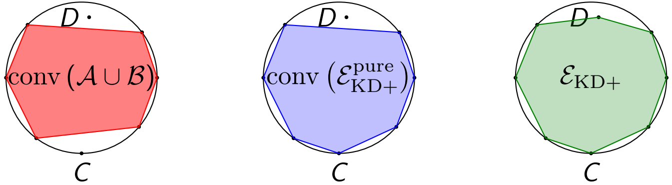


- Two bases: $\mathcal{A}$= eigenvectors of J_3 and $\mathcal{B}$ = eigenvectors of $\vec{n} \cdot \vec{J}$ with $||\vec{n}||_2 = 1$.

$$\mathcal{A} \cup \mathcal{B} \subsetneq \mathcal{E}_{\mathrm{KD+}}.$$

- There exist at least one point like C or D !

Main results: the case $d = 3$, spin 1



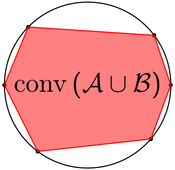
- $\mathcal{A}$= eigenvectors of J_3 and $\mathcal{B}$ = eigenvectors of $\vec{n} \cdot \vec{J}$ with $\vec{n} = \frac{1}{3}\begin{pmatrix} 2 & 2 & -1 \end{pmatrix}$.

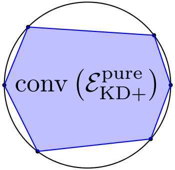
$$\mathrm{conv}\,(\mathcal{A} \cup \mathcal{B}) \subsetneq \mathrm{conv}\left(\mathcal{E}_{\mathrm{KD}+}^{\mathrm{pure}}\right) \subsetneq \mathcal{E}_{\mathrm{KD}+}.$$

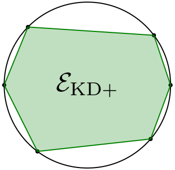
- There exist extreme states of $\mathcal{E}_{\mathrm{KD}+}$ that are **mixed** !
- There exist states that are KD positive BUT **all their convex decompositions contain at least one pure state that is not KD positive.**

- This situation also occurs in dimension $d = 2^n$ for all $n > 2$.

Main results: DFT in dimension $d \geqslant 2$







Theorem (C.L, D.R.M. Arvidsson-Shukur, S. De Bièvre, arXiv June 2023)

The equality

$$\mathcal{E}_{\mathrm{KD+}} = \mathrm{conv}\,(\mathcal{A} \cup \mathcal{B})$$

holds if the transition matrix is the DFT matrix ***in prime dimension****.*

- For non prime dimensions, $\mathcal{A} \cup \mathcal{B} \subsetneq \mathcal{E}_{\mathrm{KD+}}^{\mathrm{pure}}$ [J. Xu, arXiv 2023].
- Question for non prime dimensions: $\mathcal{E}_{\mathrm{KD+}} = \mathrm{conv}\left(\mathcal{E}_{\mathrm{KD+}}^{\mathrm{pure}}\right)$?

Take home messages

What did we do ?

- Identified situations where $\mathcal{E}_{\mathrm{KD+}} = \mathrm{conv}\,(\mathcal{A} \cup \mathcal{B})$: problem solved !

- Found KD-positive states that cannot be decomposed as convex combinations of KD-positive pure states.

What is left to do ?

- Conjecture: $\mathcal{E}_{\mathrm{KD+}} = \mathrm{conv}\,(\mathcal{A} \cup \mathcal{B})$ for a set of probability 1 in all dimensions.

- Need to construct witnesses and measures for KD-positivity.
  Work in progress.

THANK YOU FOR YOUR ATTENTION !