Geometry of the Kirkwood-Dirac-positive states

Christopher Langrenez
joint work with David R. M. Arvidsson-Shukur & Stephan De Bièvre
arXiv:2306.00086

QUIDIQUA, Lille, France
10 November 2023
The Kirkwood-Dirac (KD) distribution: definition

- $\mathcal{H}$: Hilbert space of finite dimension $d$. $(|a_i\rangle)_{i \in [1,d]}$ and $(|b_j\rangle)_{j \in [1,d]}$ two orthonormal bases of $\mathcal{H}$. $U_{ij} = \langle a_i|b_j\rangle$ transition matrix.
- Examples:
  - $d = 2$, $(|0\rangle, |1\rangle)$ and $(|+\rangle, |-\rangle)$;
  - $d \geq 2$, think of $U$ as Discrete Fourier Transform (DFT).
- The Kirkwood(1933)-Dirac(1945) quasiprobability distribution of $\rho$:

\[
\forall (i,j) \in [1,d]^2, \quad Q_{ij}(\rho) = \langle b_j|a_i\rangle\langle a_i|\rho|b_j\rangle \in \mathbb{C}.
\]

\[
\sum_{i=1}^{d} Q_{ij}(\rho) = \langle b_j|\rho|b_j\rangle, \quad \sum_{j=1}^{d} Q_{ij}(\rho) = \langle a_i|\rho|a_i\rangle \quad \text{and} \quad \sum_{i,j} Q_{ij}(\rho) = 1.
\]

- Analogous to the Wigner function. However, more flexible as not designed only for position and momentum.
KD positivity

- \( \rho \) is a KD-positive state if \( Q_{ij}(\rho) \geq 0 \) for all \((i, j) \in [1, d]^2\).

Examples: \( Q_{kl}(|a_1\rangle\langle a_1|) = \delta_{1,k} |\langle a_1|b_1\rangle|^2 \). \( |a_1\rangle\langle a_1| \) KD-positive.

\( A = \{|a_i\rangle\langle a_i| \mid i \in [1, d]\} \) and \( B = \{|b_j\rangle\langle b_j| \mid j \in [1, d]\} \) KD-positive.

\[ E_{KD+} \text{ is the set of KD-positive states.} \]

- \( E_{KD+} \) is convex: if \( \rho_1, \rho_2 \in E_{KD+} \), for \( \lambda \in [0, 1] \), \( \lambda \rho_1 + (1 - \lambda) \rho_2 \in E_{KD+} \).

**GOAL:** obtain a precise/complete description of the geometry of \( E_{KD+} \).

- KD-nonpositive states linked with quantum advantages
• A convex set: the segment between two points in the set lies in the set.
• A convex set: the segment between two points in the set lies in the set.
• A convex set: The segment between two points in the set lies in the set.

• $\rho$ is extreme: if $\rho = \lambda \rho_1 + (1 - \lambda) \rho_2$ then $\rho_1 = \rho$ and $\rho_2 = \rho$.

**Theorem (Krein-Milman)**

$K$ is convex, closed and bounded $\Rightarrow K = \text{conv}(K^{\text{ext}})$

where $K^{\text{ext}}$ is the set of extreme points of $K$ and $\text{conv}()$ is the convex hull.
A convex set: The segment between two points in the set lies in the set.

\[
\text{conv} (K_{\text{ext}}) = \bigcup_{n \in \mathbb{N}^*} \left\{ \sum_{p=1}^{n} \lambda_p \rho_p, \rho_p \in K_{\text{ext}}, \sum_{p=1}^{n} \lambda_p = 1, \lambda_p \geq 0 \right\}
\]

The decomposition may not be unique!
Looking for the extreme points of $\mathcal{E}_{\text{KD}+}$ (1)

- KD distribution: $Q_{ij}(\rho) = \langle b_j|a_i\rangle\langle a_i|\rho|b_j\rangle$.
- The extreme points of the set of quantum states $= \text{the pure states}$.
- **FACT:** Pure KD positive states are extreme states of $\mathcal{E}_{\text{KD}+}$.
- Denote by $\mathcal{E}_{\text{KD}+}^{\text{pure}}$ the set of pure KD positive states.
- $A = \{|a_i\rangle\langle a_i| : i \in [1, d]\} \subset \mathcal{E}_{\text{KD}+}^{\text{pure}}$; $B = \{|b_j\rangle\langle b_j| : j \in [1, d]\} \subset \mathcal{E}_{\text{KD}+}^{\text{pure}}$.
- $\mathcal{A} \cup \mathcal{B} \subsetneq \mathcal{E}_{\text{KD}+}^{\text{pure}} \subsetneq \mathcal{E}_{\text{KD}+}^{\text{ext}}$.
- **Difficulty:** the space of quantum states has dimension $d^2$ where $d = \dim(\mathcal{H})$. 
Looking for the extreme points of $\mathcal{E}_{KD+}$ (2)

\[\text{conv}(A \cup B) \subsetneq \text{conv}(\mathcal{E}_{KD+}^{\text{pure}}) \subsetneq \mathcal{E}_{KD+} = \text{conv}(\mathcal{E}_{KD+}^{\text{ext}}).\]

- **Difficulty 1**: Find or rule out the existence of pure KD-positive states that are not basis states (points like C).
- **Difficulty 2**: Find or rule out the existence of mixed KD-positive states (points like D).
Looking for extreme points of $\mathcal{E}_{\text{KD}+} (3)$

$$\text{conv} \left( \mathcal{A} \cup \mathcal{B} \right) \subsetneq \text{conv} \left( \mathcal{E}_{\text{KD}+}^{\text{pure}} \right) \subsetneq \mathcal{E}_{\text{KD}+} = \text{conv} \left( \mathcal{E}_{\text{KD}+}^{\text{ext}} \right).$$

- **Difficulty 1**: Find or rule out the existence of pure KD-positive states that are not basis states (points like C).
- **Difficulty 2**: Find or rule out the existence of mixed extreme KD-positive states (points like D).
Main results: the case $d = 2$ (qubits)

If $U_{ij} = \langle a_i | b_j \rangle \neq 0$ for all $(i, j) \in [1, d]$

\[ A = \{|0\rangle\langle 0|, |1\rangle\langle 1|\} \]
\[ B = \{|+\rangle\langle +|, |--\rangle\langle --|\} \]

\[ \text{conv} (A \cup B) = E_{KD^+}. \]
Main results: the case \( d = 3 \), random bases


The equality

\[
\mathcal{E}_{\text{KD}+} = \text{conv} (\mathcal{A} \cup \mathcal{B})
\]

holds for an open dense set of unitary matrices in dimension 3.

- Take two random bases \( \mathcal{A} \) and \( \mathcal{B} \) (choose a random unitary matrix);
- With probability 1, one then has \( \mathcal{E}_{\text{KD}+} = \text{conv} (\mathcal{A} \cup \mathcal{B}) \).
Main results: the case $d = 3$, spin $1$

- Two bases: $\mathcal{A}$ = eigenvectors of $J_3$ and $\mathcal{B}$ = eigenvectors of $\vec{n} \cdot \vec{J}$ with $||\vec{n}||_2 = 1$.

\[ \mathcal{A} \cup \mathcal{B} \subsetneq \mathcal{E}_{KD+}. \]

- There exist at least one point like C or D!
Main results: the case $d = 3$, spin 1

- $\mathcal{A} = $ eigenvectors of $J_3$ and $\mathcal{B} = $ eigenvectors of $\vec{n} \cdot \vec{J}$ with $\vec{n} = \frac{1}{3} \begin{pmatrix} 2 & 2 & -1 \end{pmatrix}$.

- $\text{conv}(\mathcal{A} \cup \mathcal{B}) \subsetneq \text{conv}(\mathcal{E}_{\text{KD+}}^\text{pure}) \subsetneq \mathcal{E}_{\text{KD+}}$.

- There exist extreme states of $\mathcal{E}_{\text{KD+}}$ that are mixed!
- There exist states that are KD positive but all their convex decompositions contain at least one pure state that is not KD positive.

- This situation also occurs in dimension $d = 2^n$ for all $n > 2$. 
Main results: DFT in dimension \( d \geq 2 \)

\[
\text{conv} \left( \mathcal{A} \cup \mathcal{B} \right) \quad \text{conv} \left( \mathcal{E}_{\text{pure}}^{\text{KD+}} \right) \quad \mathcal{E}_{\text{KD+}}
\]


The equality

\[
\mathcal{E}_{\text{KD+}} = \text{conv} \left( \mathcal{A} \cup \mathcal{B} \right)
\]

holds if the transition matrix is the DFT matrix in prime dimension.

- For non prime dimensions, \( \mathcal{A} \cup \mathcal{B} \not\subseteq \mathcal{E}_{\text{KD+}}^{\text{pure}} \) [J. Xu, arXiv 2023].
- Question for non prime dimensions: \( \mathcal{E}_{\text{KD+}} = \text{conv} \left( \mathcal{E}_{\text{KD+}}^{\text{pure}} \right) \)?
Take home messages

What did we do?

• Identified situations where $\mathcal{E}_{KD+} = \text{conv} (\mathcal{A} \cup \mathcal{B})$: problem solved!

• Found KD-positive states that cannot be decomposed as convex combinations of KD-positive pure states.

What is left to do?

• Conjecture: $\mathcal{E}_{KD+} = \text{conv} (\mathcal{A} \cup \mathcal{B})$ for a set of probability 1 in all dimensions.

• Need to construct witnesses and measures for KD-positivity.
  Work in progress.
THANK YOU FOR YOUR ATTENTION!