Quantum Measurements with Quasiprobabilities

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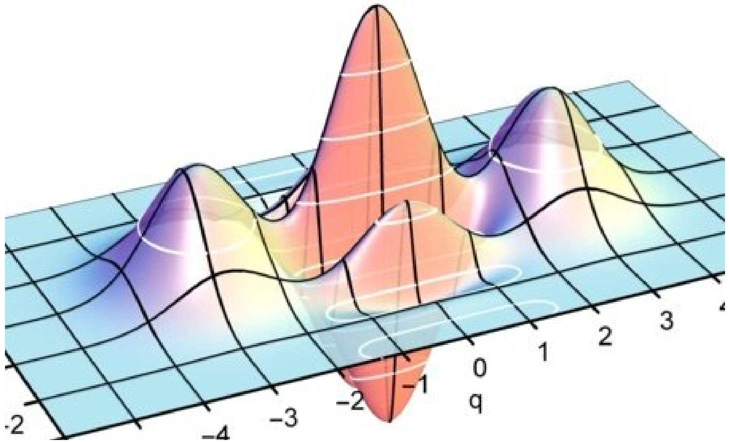
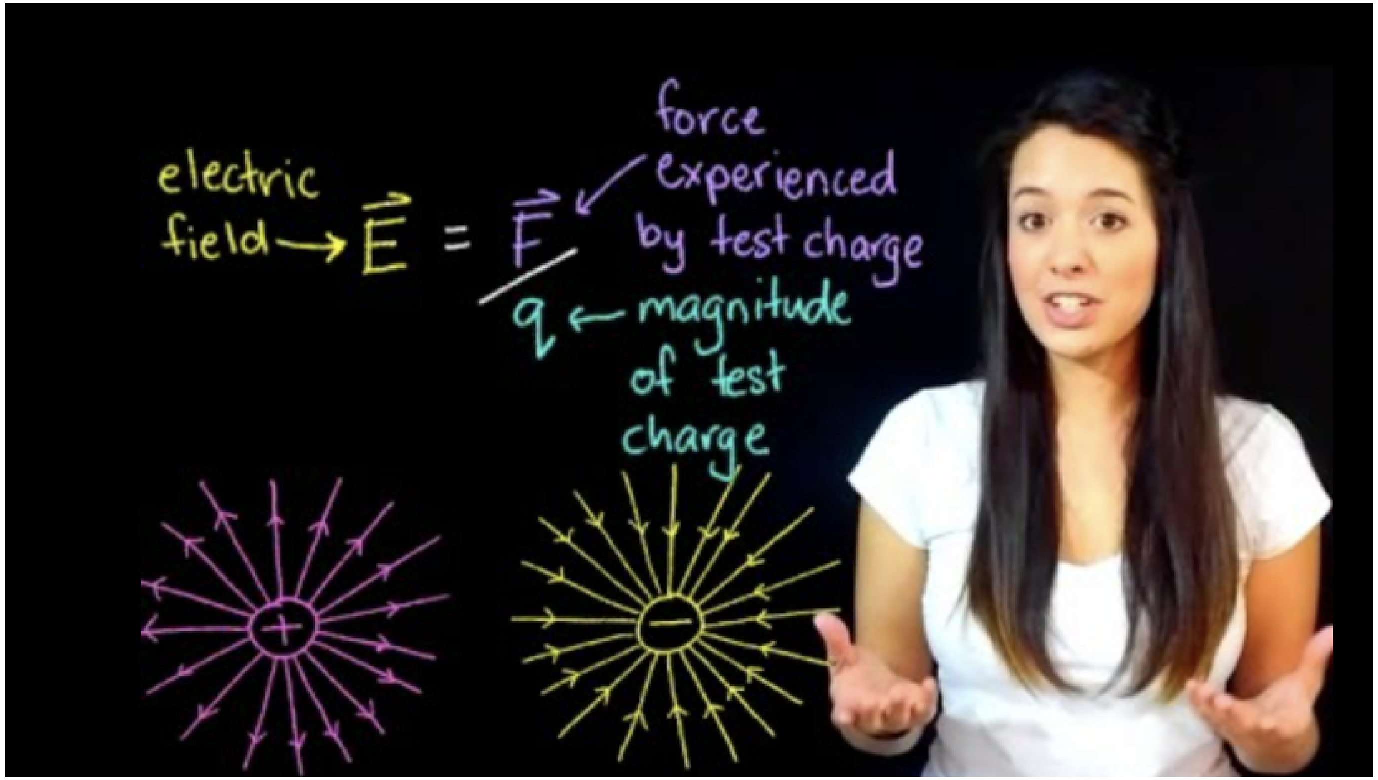


Classical Field Theory



How do we **measure** what the Electric Field is at some point x in space?

We put a "**test charge**" at x and look at its *response*.

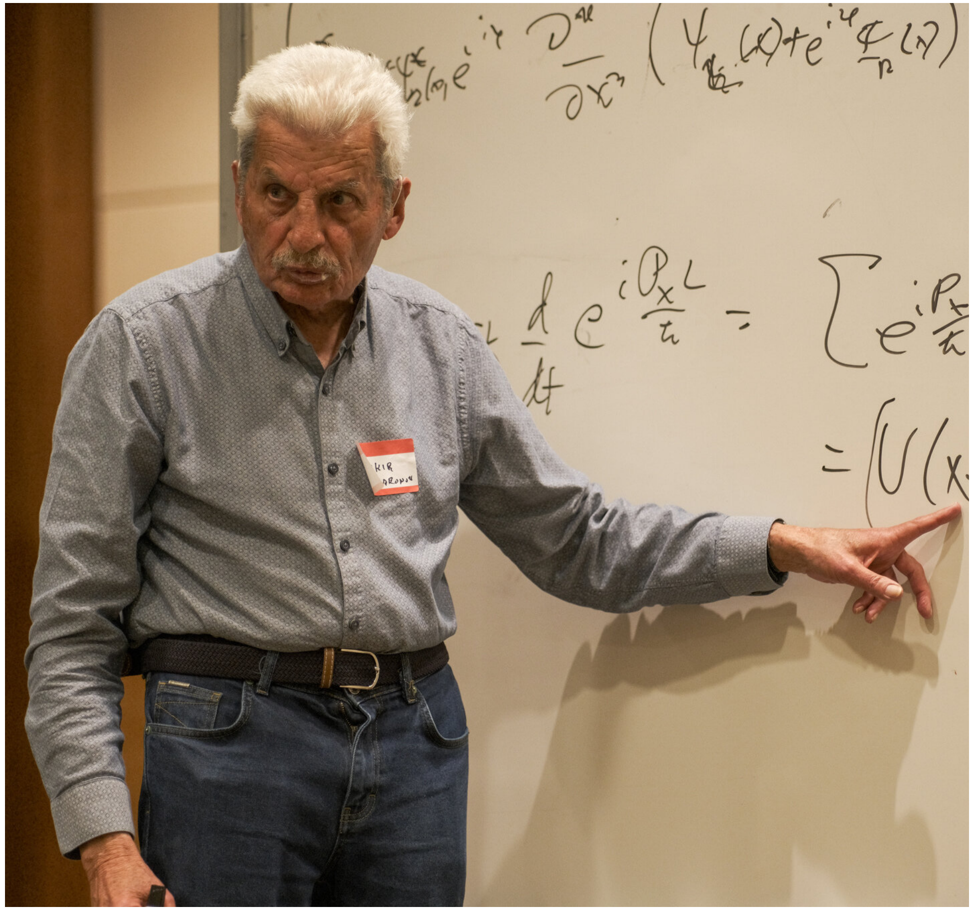
Electric Field: $\vec{E} \equiv \left.\frac{\vec{F}}{q}\right|_{q\to 0}$

We take the **weak interaction limit** where *the test charge is small*, so the field we are measuring is **not disturbed** by the test charge itself.

Quantum Theory

Yakir Aharonov:

"Let's do *exactly the same thing* with quantum theory."



Probe charge: $|\phi\rangle$ s.t. $\langle\hat{x}\rangle = 0$

System: $|\psi\rangle$

Interaction: $\hat{H} = q\hat{A} \otimes \hat{p}$

Response:

$$e^{-iqt\hat{A}\hat{p}/\hbar}|\psi\rangle|\phi\rangle = \sum_a \int dx|a\rangle|x\rangle\,\langle a|\psi\rangle\,\langle x - qat|\phi\rangle$$

Mean velocity per charge yields:

$$\left.\frac{\partial_t\langle\hat{x}\rangle}{q}\right|_{q\to 0} = \sum_a |\langle a|\psi\rangle|^2 a = \langle\hat{A}\rangle$$

Expectation value of $\hat{A}$ measured *without probe disturbing the system (much).*

Postselecting Quantum Theory

Yakir Aharonov:

"Let's do *exactly the same thing* with quantum theory, but also *add a final postselection measurement*."

System Postselection: $\langle f|$

Small test charge response:

$$\langle f|e^{-iqt\hat{A}\hat{p}/\hbar}|\psi\rangle|\phi\rangle \approx \langle f|\psi\rangle(1 - iqtA_w\hat{p}/\hbar)|\phi\rangle$$
$$\approx \langle f|\psi\rangle \int dx|x\rangle\ \langle x - qA_w t|\phi\rangle$$

Mean velocity per charge yields:

$$\left.\frac{\partial_t\langle\hat{x}\rangle}{q}\right|_{q\to 0} = \mathrm{Re}A_w \equiv \mathrm{Re}\frac{\langle f|\hat{A}|\psi\rangle}{\langle f|\psi\rangle}$$

The expectation value of $\hat{A}$ *conditioned on a final postselection* $\langle f|$ is a **weak value**.

Aharonov et al. PRL 1988

Weak Value:

A "conditioned expectation value" measured in the *weak interaction* ("test charge") limit.

$$A_w \equiv \frac{\langle f|\hat{A}|\psi\rangle}{\langle f|\psi\rangle} = \frac{\langle \psi|f\rangle\langle f|\hat{A}|\psi\rangle}{|\langle f|\psi\rangle|^2}$$

An expectation value is partitioned into a convex mixture of weak values,
each specific to a particular "postselection".

$$\langle A\rangle \equiv \langle\psi|\hat{A}|\psi\rangle = \sum_f \langle\psi|f\rangle\langle f|\hat{A}|\psi\rangle = \sum_f |\langle f|\psi\rangle|^2 \frac{\langle f|\hat{A}|\psi\rangle}{\langle f|\psi\rangle} = \sum_f P_{f|\psi}\, A_{w,f}$$

Weak values are *not necessarily constrained* by the spectrum of the operator $\hat{A}$.

Kirkwood-Dirac Quasiprobabilities:

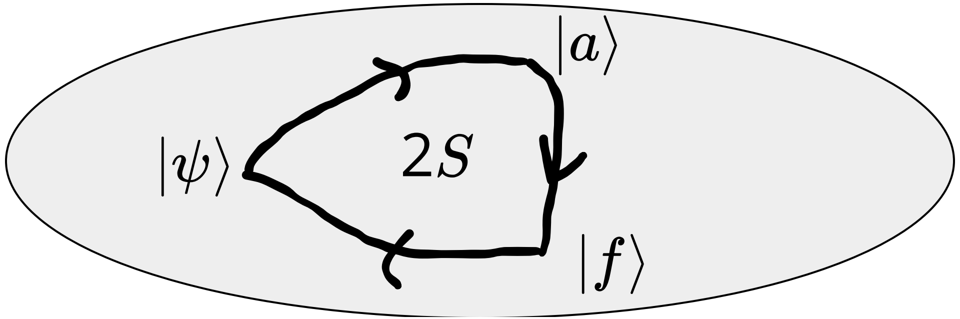
The "conditioned probabilities" that average the eigenvalues of $\hat{A}$ are *quasiprobabilities*.

$$A_w \equiv \frac{\langle\psi|f\rangle\langle f|\hat{A}|\psi\rangle}{|\langle f|\psi\rangle|^2} = \sum_a a \frac{\langle\psi|f\rangle\langle f|a\rangle\langle a|\psi\rangle}{|\langle f|\psi\rangle|^2} = \sum_a a \frac{Q_{a,f|\psi}}{P_{f|\psi}}$$

The Kirkwood-Dirac quasiprobabilities are *conditioned* by the postselection likelihoods.

$$Q_{a,f|\psi} \equiv \langle\psi|f\rangle\langle f|a\rangle\langle a|\psi\rangle = |Q_{a,f|\psi}|\, e^{iS}$$

The phase is a **geometric Berry/Pancharatnam phase** determined by the *oriented area* enclosed by *geodesics* connecting three quantum states.



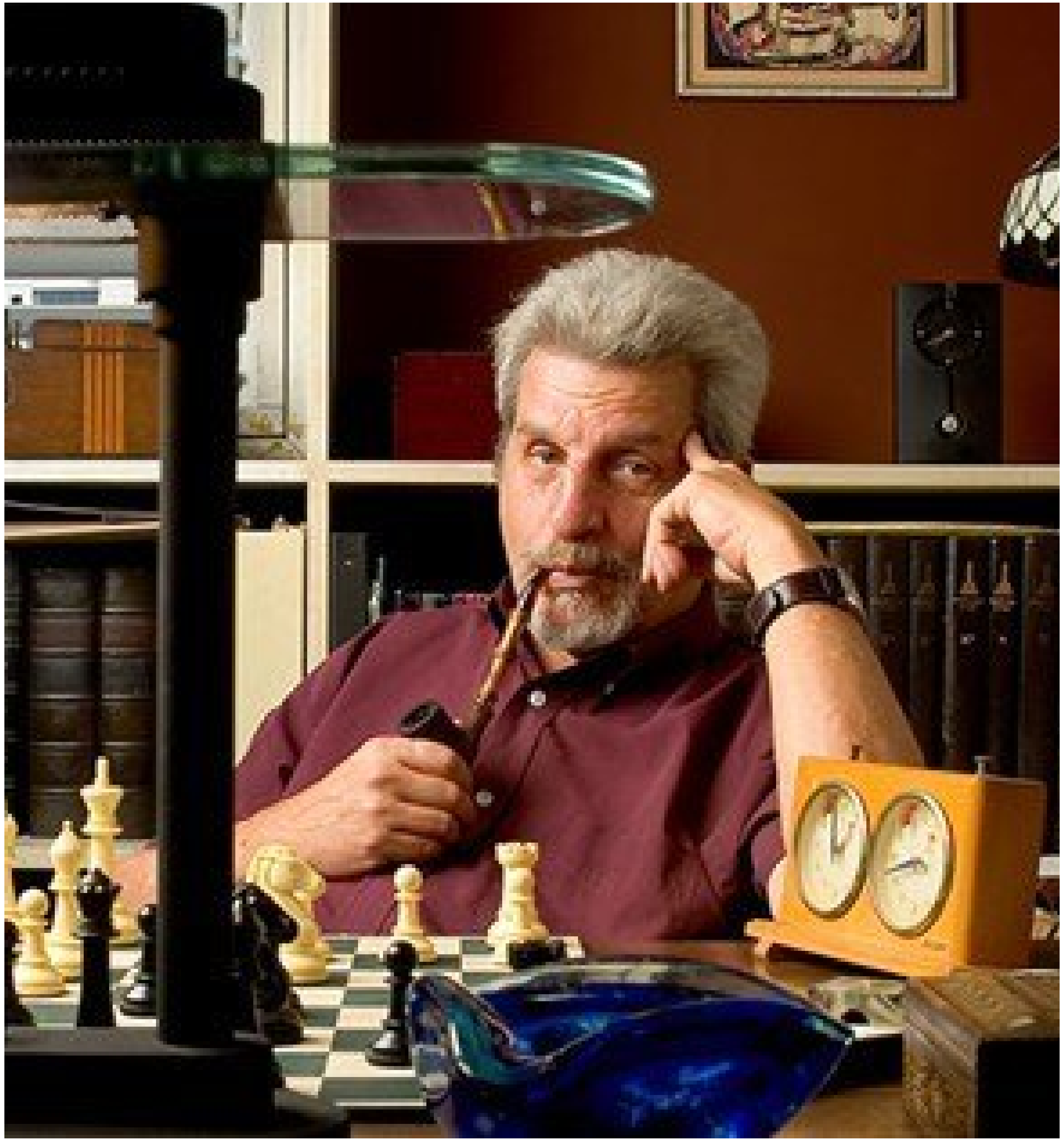
Classically compatible states enclose zero area.
Non-zero phases encode *dynamical* incompatibility and a *temporal directionality* for closing the loop.

Hofmann, NJP 2011

Dynamical Weak Values

Due to their close connection to the geometry of quantum state space, weak values (and thus the KD QP) can appear **even without making any explicit weak measurements**.

Let's explore several examples.

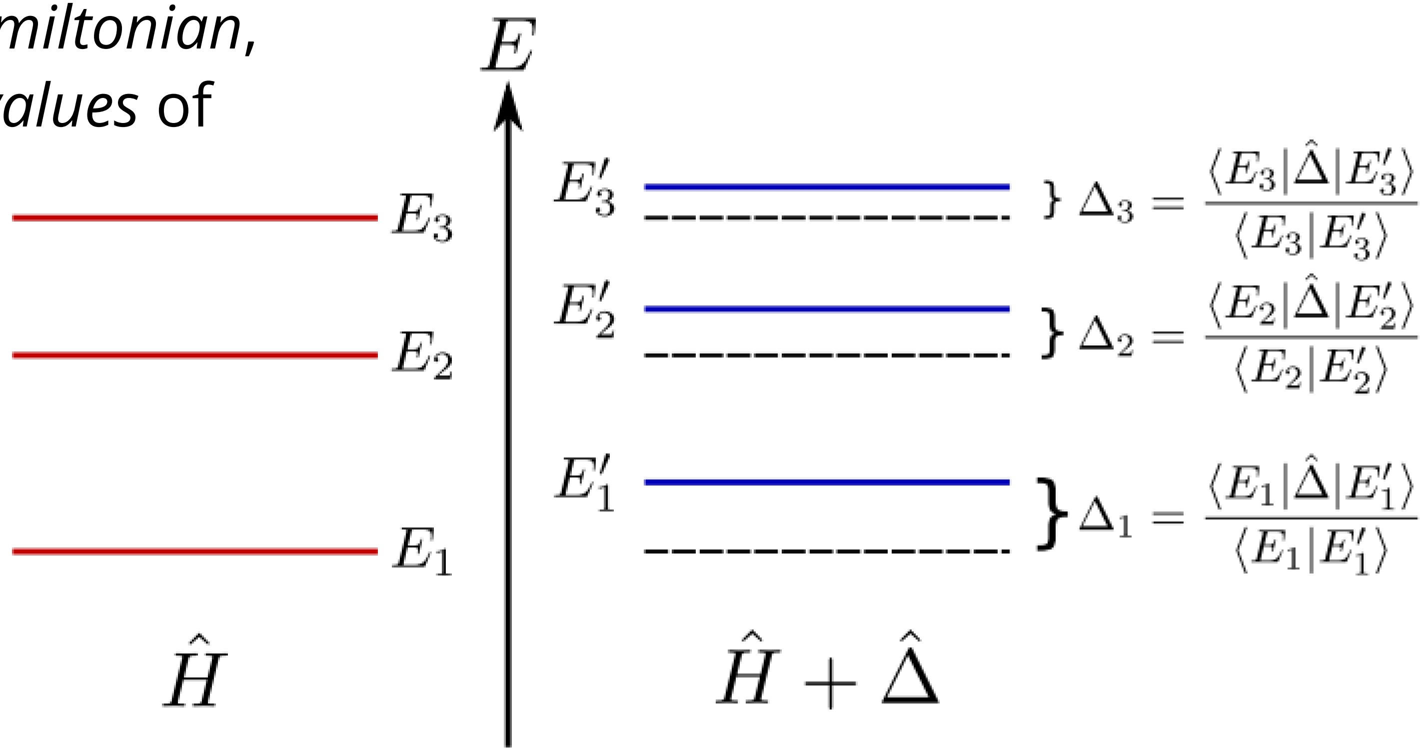


Trivial Example of Dynamical Weak Values:

When a *perturbation* is added to a *Hamiltonian*, the energy spectra will shift by *weak values* of the perturbation.

$$\hat{H}|E_k\rangle = E_k|E_k\rangle$$

$$(\hat{H}+\hat{\Delta})|E'_j\rangle = E'_j|E'_j\rangle$$



$$\langle E'_j|(\hat{H}+\hat{\Delta})|E_k\rangle = E'_j\langle E'_j|E_k\rangle = E_k\langle E'_j|E_k\rangle + \langle E'_j|\hat{\Delta}|E_k\rangle$$

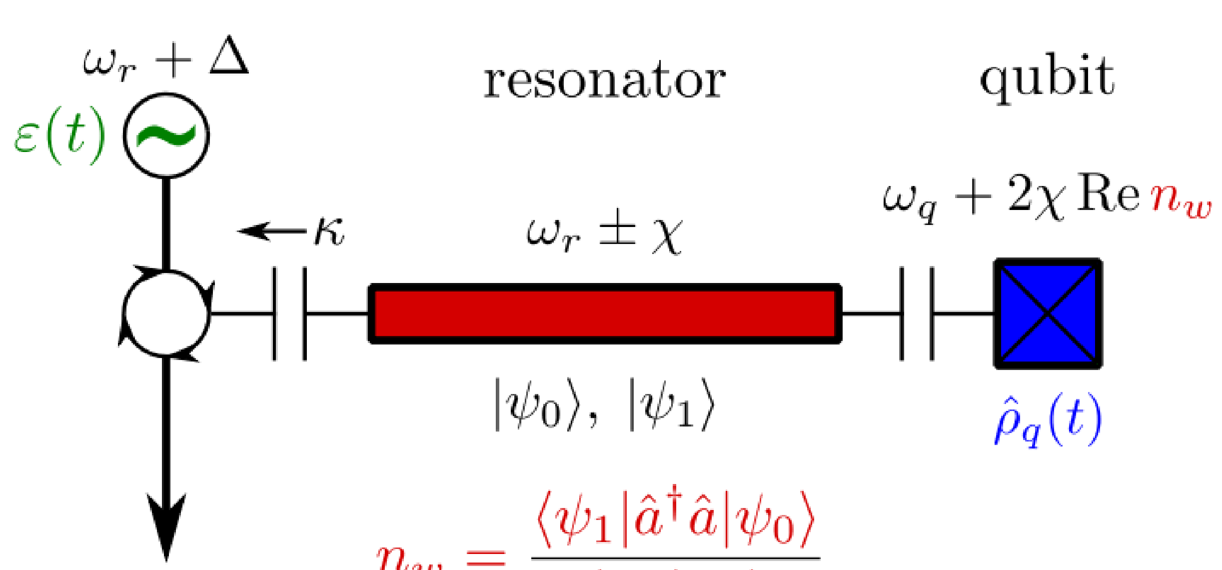
$$E'_j - E_k = \frac{\langle E'_j|\hat{\Delta}|E_k\rangle}{\langle E'_j|E_k\rangle}$$

Measurable energy shifts caused by a perturbation are always (purely real) weak values.

"Strange weak values" outside the spectrum of the perturbation are *very common*.

JD, PRA 91 032116 (2015)

Nontrivial Example: Circuit QED



$$n_w = \frac{\langle\psi_1|\hat{a}^\dagger\hat{a}|\psi_0\rangle}{\langle\psi_1|\psi_0\rangle}$$

For pump ε and decay rate κ, the steady states yield:

$$n_w \approx \frac{4\varepsilon^2}{\kappa^2}\left[1 + i\frac{4\chi}{\kappa}\right] \equiv \bar{n} + i\frac{4\chi\bar{n}}{\kappa}$$

- Real part : **AC Stark shift** $\quad \Delta\omega_q = 2\chi\,\mathrm{Re}\,n_w = 2\chi\bar{n}$
- Imaginary part : **ensemble dephasing** $\quad \Gamma = 2\chi\,\mathrm{Im}\,n_w = \frac{8\chi^2\bar{n}}{\kappa}$

Dispersive Coupling Hamiltonian:

$$\hat{H} = \frac{\omega_q}{2}\hat{\sigma}_z + \omega_r\hat{a}^\dagger\hat{a} + \chi\hat{\sigma}_z\hat{a}^\dagger\hat{a}$$

At **steady state**, the balance of pump and decay from the resonator leaves the qubit entangled with distinct coherent states in the resonator:

$$|\Psi\rangle = c_0|0\rangle|\psi_0\rangle + c_1|1\rangle|\psi_1\rangle$$

The **coherence** of the **reduced qubit state** is thus:

$$\rho_{01}(t) = c_1^* c_0\langle\psi_1|\psi_0\rangle$$

This **reduced state** qubit coherence evolves as:

$$\partial_t\rho_{01}(t) = i[\omega_q + 2\chi\,n_w]\rho_{01}(t)$$

↑ Photon number **weak value**!

JD, PRA 91 032116 (2015)

Nontrivial example: Hamilton-Jacobi Theory

Schrodinger Equation: $$i\hbar\partial_t|\psi(t)\rangle = \left[\frac{\hat{p}^2}{2m} + V(\hat{x})\right]|\psi(t)\rangle$$

Hamilton's Principle Function : $$S(t,x) \equiv -i\hbar\ln\langle x|\psi(t)\rangle$$

Momentum *defined in the usual way* is the **weak value** of the momentum operator:

$$p_w(t,x) \equiv \partial_x S(t,x) = \frac{-i\hbar\partial_x\psi(t,x)}{\psi(t,x)} = \frac{\langle x|\hat{p}|\psi(t)\rangle}{\langle x|\psi(t)\rangle}$$

Real part: Bohmian momentum
Imaginary part: Nelson Osmotic momentum

Schrodinger's Equation is equivalent to a **quantum Hamilton-Jacobi Equation**:

$$\partial_t S(t,x) + H_w[t,x,p_w(t,x)] = 0$$

$$H_w[t,x,p_w(t,x)] = \frac{\langle x|\hat{p}^2/2m + V(\hat{x})|\psi\rangle}{\langle x|\psi\rangle}$$

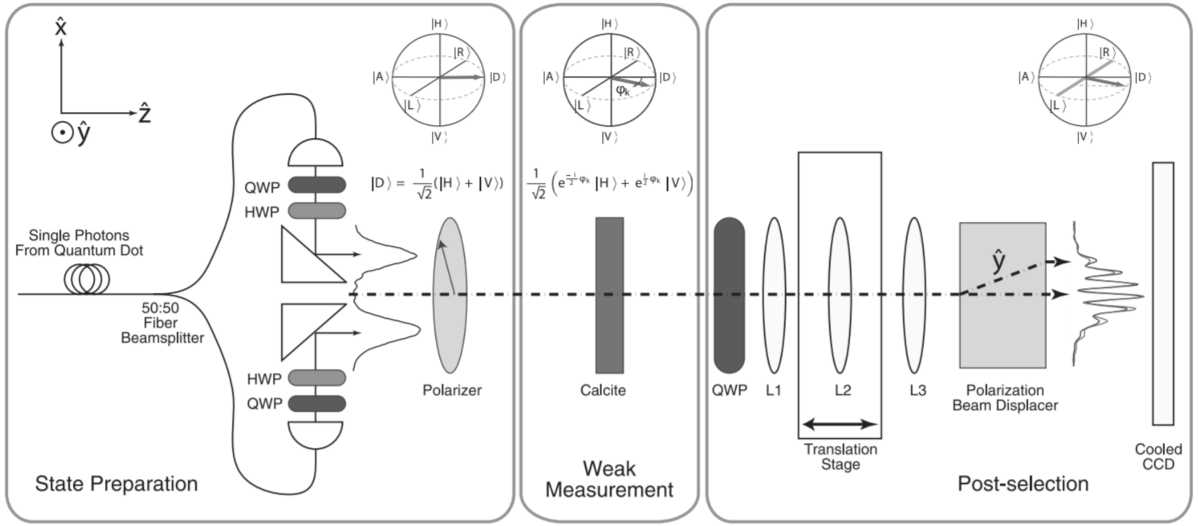
Imaginary part is a *continuity equation* for probability. Real part is *classical* HJ Equation:

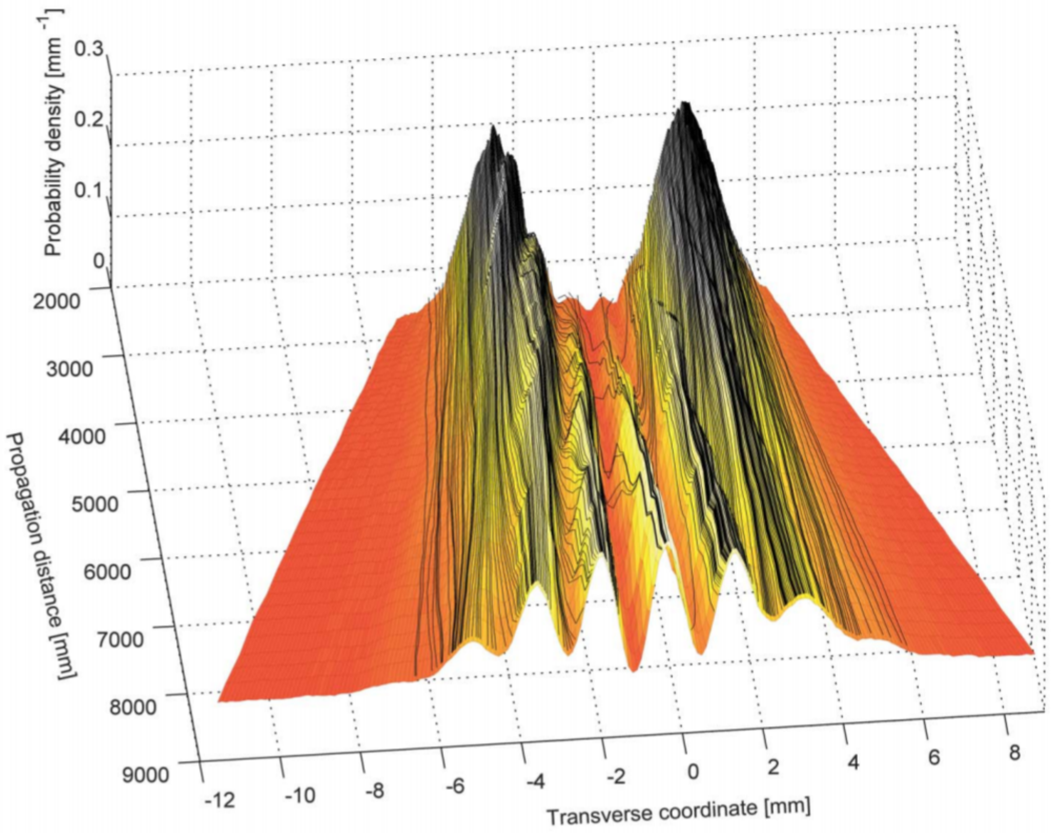
$$\mathrm{Re}H_w[t,x,p(t,x)] = \frac{(\mathrm{Re}\,p_w(t,x))^2}{2m} + V(x) + Q(x)$$

"Quantum Potential": *Weak Variance* of momentum away from mean weak value

$$Q(x) = \frac{\langle x|(\hat{p} - \mathrm{Re}\,p_w(t,x))^2|\psi(t)\rangle}{\langle x|\psi(t)\rangle} = -\frac{\hbar^2}{2m}\frac{\partial_x^2|\psi(x,t)|}{|\psi(x,t)|}$$

Nontrivial example: Classical field streamlines





Weak values also appear as **physical properties** of a classical field, even when there is not an obvious "weak measurement" at the level of individual field quanta.

$$\mathrm{Re}\,\mathbf{p}(\mathbf{r}) = \mathrm{Re}\frac{\langle x|\hat{p}|\psi\rangle}{\langle x|\psi\rangle} = \frac{\omega}{c^2}\frac{S_o(\mathbf{r})}{W(\mathbf{r})}$$

Momentum weak value proportional to the local (orbital part of the) **Poynting vector** S_O of an optical field, scaled by the frequency ω and energy density W.

This can be *measured* and used to reconstruct "averaged trajectories" for the mean momentum streamlines.

The average local momentum corresponds to the local *optical pressure* felt by *small probe particles,* and thus the momentum part of the *stress-energy tensor.*

Kocsis et al., Science (2011)
Bliokh et al. NJP (2013)

Nontrivial Example: Bessel beams

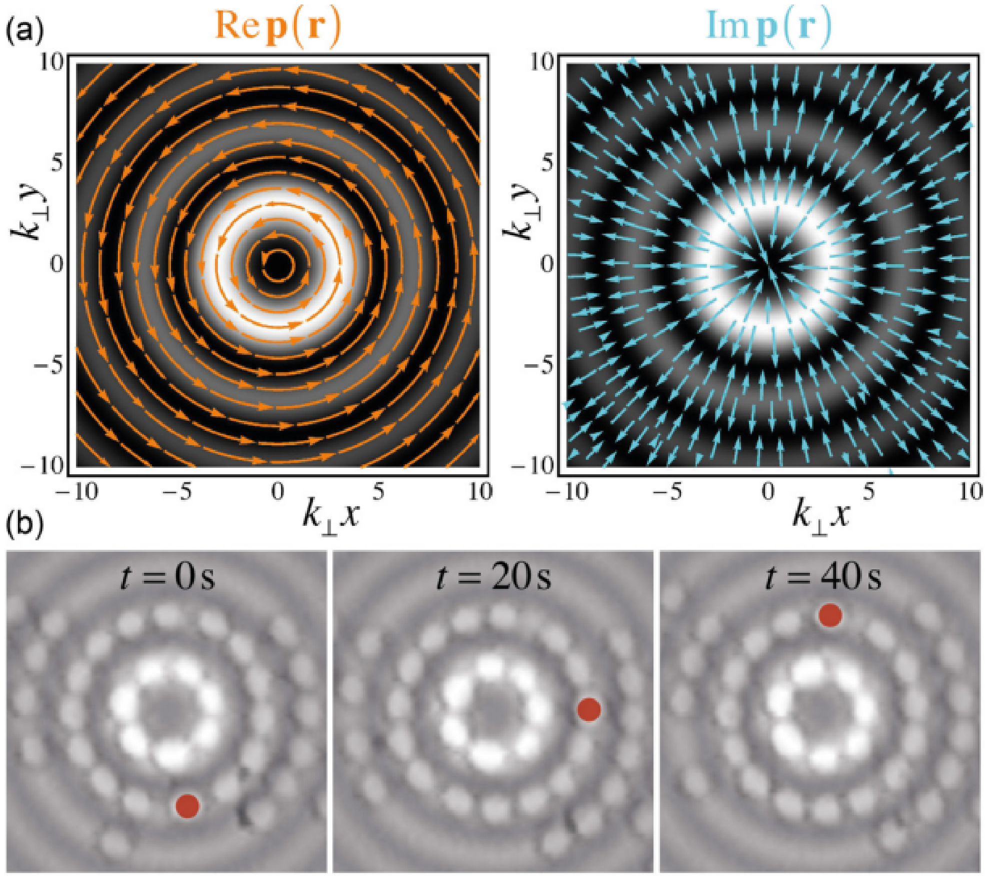


Figure 3. (a) Intensity (grayscale background) and stream distributions of the real and imaginary parts of the normalized local momentum $\mathbf{p}(\mathbf{r})$, equations (9)–(11), in the transverse plane of the Bessel beam (17) with $\ell = 2$. (b) Successive frames image the motion of the probe particles in the Bessel-beam field (taken from [19a] and one particle is marked by the red circle). A movie of this motion, available in [19a], shows that the inner rings move faster because the radiation force is proportional to the field intensity and to $1/r$. The particles are trapped at the radial field maxima because of the gradient force anti-parallel to the 'osmotic velocity' $\mathrm{Im}\,\mathbf{p}(\mathbf{r})$.

Both *real* and *imaginary* parts of this local momentum average also describe physical properties of classical "vortex beams" like Bessel beams.

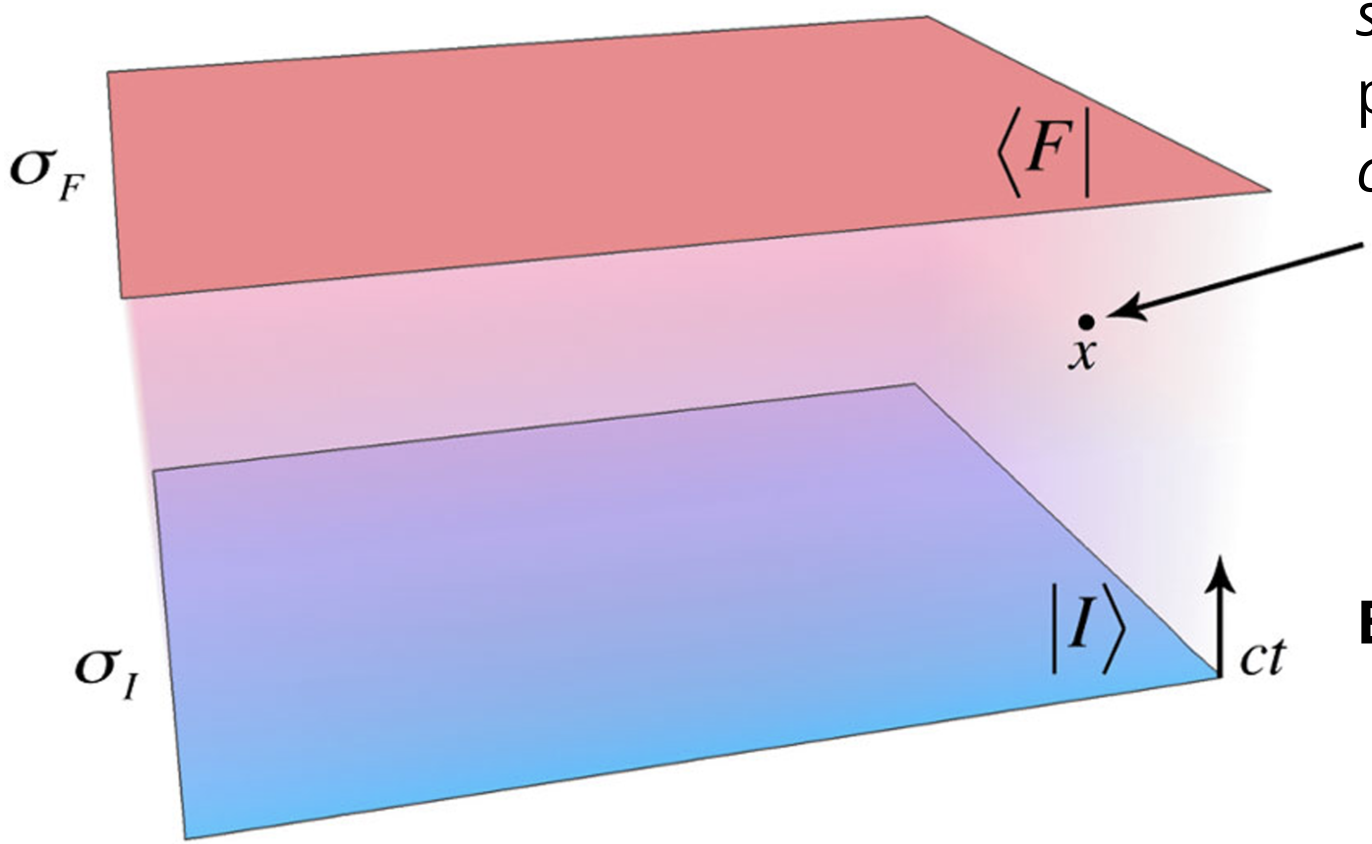
$$\mathbf{p}_w(\mathbf{r}) = \frac{\langle x|\hat{p}|\psi\rangle}{\langle x|\psi\rangle}$$

The **real part** of the momentum weak value appears as the circulating local *orbital momentum* that can be transferred to probe particles by pushing them around in circular orbits.

The **imaginary part** is directed *radially* and *confines* the optical intensity into concentric rings, similarly trapping probe particles to orbit only within the optical rings.

Bliokh et al., NJP (2013)

Quantum Field Theory



But wait, there's more!

Schwinger taught us to do *exactly the same thing in quantum field theory* to probe *mean (classical)* fields and their *correlations*

"Test charge" current:

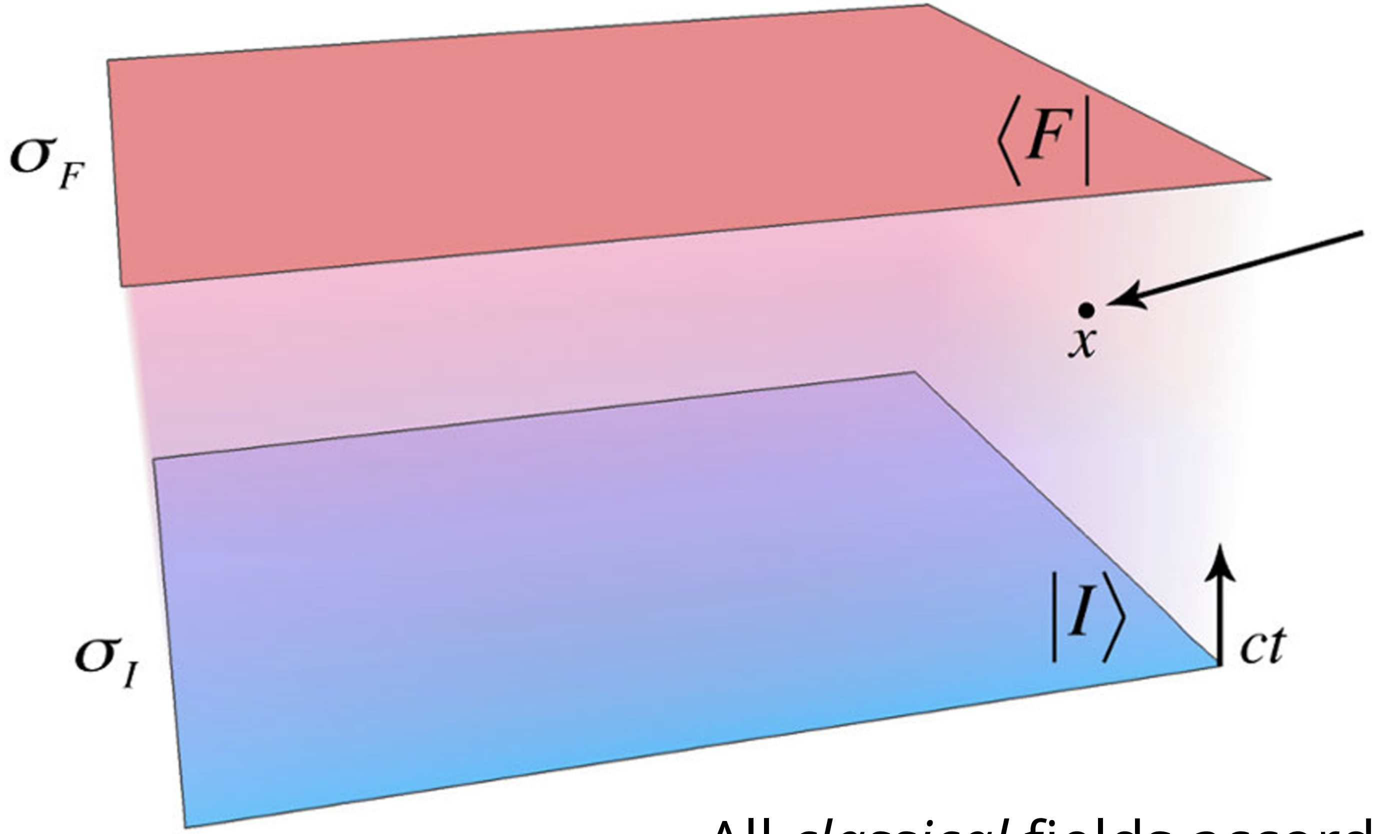
- Intermediate time t: $J(x)$
- Take limit as $J \to 0$

Boundary conditions:

- Past time t_I: $|I\rangle$
- Future time t_F: $\langle F|$

Schwinger, PR (1951)

QFT: Classical Mean Fields



Generating Functional:

$$W[J] = -i\hbar \ln\langle F|\hat{U}[J]|I\rangle$$

Effective Action:

$$\Gamma[\varphi] = W[J] - \int d^4x J(x)\varphi(x)$$

"Classical" Field:

$$\varphi(x) \equiv \left.\frac{\delta W[J]}{\delta J(x)}\right|_{J\to 0} = \frac{\langle F(t)|\hat{\varphi}(x)|I(t)\rangle}{\langle F(t)|I(t)\rangle}$$

"Classical" Equation of Motion:

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} = -J(x) \to 0$$

All *classical* fields according to QFT are **weak values** of the field operators.

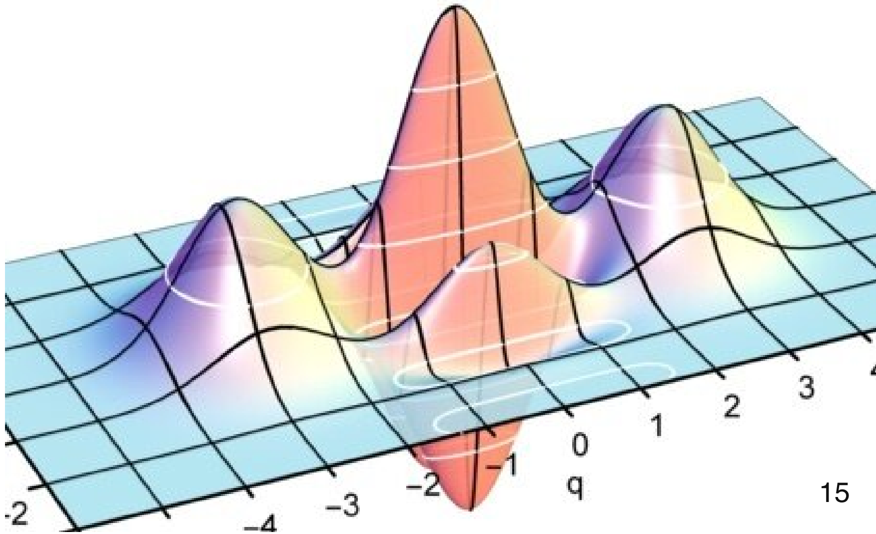
JD, PRL (2014)

Summary

Weak values are prevalent in the quantum formalism,
even without performing weak measurements,
which means that *Kirkwood-Dirac quasiprobabilities* are also!

- Conditioned average of weakly measured observable
- Spectral shifts due to perturbations
- Ensemble-averaged dynamical parameters
- Classical mean field properties
- Classical limit for observable values
- ...and many more examples

Thank you!



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