Quantum Measurements with Quasiprobabilities

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Classical Field Theory

How do we measure what the Electric Field is at some point $x$ in space?

We put a "test charge" at $x$ and look at its response.

**Electric Field:** \[ \vec{E} \equiv \frac{\vec{F}}{q} \bigg|_{q \to 0} \]

We take the **weak interaction limit** where *the test charge is small*, so the field we are measuring is **not disturbed** by the test charge itself.
Quantum Theory

Yakir Aharonov:
"Let's do exactly the same thing with quantum theory."

Probe charge: $|\phi\rangle$ s.t. $\langle \hat{x} \rangle = 0$

System: $|\psi\rangle$

Interaction: $\hat{H} = q \hat{A} \otimes \hat{p}$

Response:
$$e^{-iqt\hat{A}\hat{p}/\hbar}|\psi\rangle|\phi\rangle = \sum_a \int dx |a\rangle|x\rangle \langle a|\psi\rangle \langle x - qat|\phi\rangle$$

Mean velocity per charge yields:
$$\frac{\partial_t \langle \hat{x} \rangle}{q} \bigg|_{q \to 0} = \sum_a |\langle a|\psi\rangle|^2 a = \langle \hat{A} \rangle$$

Expectation value of $\hat{A}$ measured without probe disturbing the system (much).
Postselecting Quantum Theory

Yakir Aharonov:
"Let's do exactly the same thing with quantum theory, but also add a final postselection measurement."

System Postselection: \( \langle f \rangle \)

Small test charge response:
\[
\langle f|e^{-iqt\hat{\mathbf{A}}\hat{\mathbf{p}}/\hbar}|\psi \rangle|\phi \rangle \approx \langle f|\psi \rangle (1 - iqtA_w\hat{\mathbf{p}}/\hbar)|\phi \rangle \\
\approx \langle f|\psi \rangle \int dx|\langle x - qA_w t|\phi \rangle |
\]

Mean velocity per charge yields:
\[
\frac{\partial_t \langle \hat{x} \rangle}{q} \bigg|_{q \to 0} = \text{Re}A_w \equiv \text{Re} \frac{\langle f|\hat{\mathbf{A}}|\psi \rangle}{\langle f|\psi \rangle}
\]

The expectation value of \( \hat{\mathbf{A}} \) conditioned on a final postselection \( \langle f \rangle \) is a \textbf{weak value}.

Aharonov et al. PRL 1988
Weak Value:

A "conditioned expectation value" measured in the *weak interaction* ("test charge") limit.

\[ A_w \equiv \frac{\langle f | \hat{A} | \psi \rangle}{\langle f | \psi \rangle} = \frac{\langle \psi | f \rangle \langle f | \hat{A} | \psi \rangle}{|\langle f | \psi \rangle|^2} \]

An expectation value is partitioned into a convex mixture of weak values, each specific to a particular "postselection".

\[ \langle A \rangle \equiv \langle \psi | \hat{A} | \psi \rangle = \sum_f \langle \psi | f \rangle \langle f | \hat{A} | \psi \rangle = \sum_f |\langle f | \psi \rangle|^2 \frac{\langle f | \hat{A} | \psi \rangle}{\langle f | \psi \rangle} = \sum_f P_{f|\psi} A_{w,f} \]

Weak values are *not necessarily constrained* by the spectrum of the operator \( \hat{A} \).
Kirkwood-Dirac Quasiprobabilities:

The "conditioned probabilities" that average the eigenvalues of $\hat{A}$ are quasiprobabilities.

$$A_w \equiv \frac{\langle \psi | f \rangle \langle f | \hat{A} | \psi \rangle}{|\langle f | \psi \rangle|^2} = \sum_a a \frac{\langle \psi | f \rangle \langle f | a \rangle \langle a | \psi \rangle}{|\langle f | \psi \rangle|^2} = \sum_a a \frac{Q_{a,f|\psi}}{P_{f|\psi}}$$

The Kirkwood-Dirac quasiprobabilities are conditioned by the postselection likelihoods.

$$Q_{a,f|\psi} \equiv \langle \psi | f \rangle \langle f | a \rangle \langle a | \psi \rangle = |Q_{a,f|\psi}| e^{iS}$$

The phase is a geometric Berry/Pancharatnam phase determined by the oriented area enclosed by geodesics connecting three quantum states.

Classically compatible states enclose zero area. Non-zero phases encode dynamical incompatibility and a temporal directionality for closing the loop.

Hofmann, NJP 2019
Dynamical Weak Values

Due to their close connection to the geometry of quantum state space, weak values (and thus the KD QP) can appear even without making any explicit weak measurements.

Let's explore several examples.
**Trivial Example of Dynamical Weak Values:**

When a *perturbation* is added to a *Hamiltonian*, the energy spectra will shift by *weak values* of the perturbation.

\[ \hat{H} |E_k\rangle = E_k |E_k\rangle \]

\[ (\hat{H} + \hat{\Delta}) |E'_j\rangle = E'_j |E'_j\rangle \]

\[ \langle E'_j | (\hat{H} + \hat{\Delta}) |E_k\rangle = E'_j \langle E'_j | E_k \rangle = E_k \langle E'_j | E_k \rangle + \langle E'_j | \hat{\Delta} | E_k \rangle \]

For each energy level:

\[ \Delta_1 = \frac{\langle E_1 | \hat{\Delta} | E'_1 \rangle}{\langle E_1 | E'_1 \rangle} \]

\[ \Delta_2 = \frac{\langle E_2 | \hat{\Delta} | E'_2 \rangle}{\langle E_2 | E'_2 \rangle} \]

\[ \Delta_3 = \frac{\langle E_3 | \hat{\Delta} | E'_3 \rangle}{\langle E_3 | E'_3 \rangle} \]

Measurable energy shifts caused by a perturbation are always (purely real) weak values.

"Strange weak values" outside the spectrum of the perturbation are very common.
Nontrivial Example: Circuit QED

\[ \hat{H} = \frac{\omega_q}{2} \hat{\sigma}_z + \omega_r \hat{a}^\dagger \hat{a} + \chi \hat{\sigma}_z \hat{a}^\dagger \hat{a} \]

At steady state, the balance of pump and decay from the resonator leaves the qubit entangled with distinct coherent states in the resonator:

\[ |\Psi\rangle = c_0 |0\rangle |\psi_0\rangle + c_1 |1\rangle |\psi_1\rangle \]

The coherence of the reduced qubit state is thus:

\[ \rho_{01}(t) = c_1^* c_0 \langle \psi_1 | \psi_0 \rangle \]

This reduced state qubit coherence evolves as:

\[ \partial_t \rho_{01}(t) = i [\omega_q + 2\chi n_{w}, \rho_{01}(t)] \]

Photon number **weak value!**

For pump \( \varepsilon \) and decay rate \( \kappa \), the steady states yield:

\[ n_{w} \approx \frac{4\varepsilon^2}{\kappa^2} \left[ 1 + i \frac{4\chi}{\kappa} \right] \equiv \bar{n} + i \frac{4\chi \bar{n}}{\kappa} \]

- Real part: **AC Stark shift**
  \[ \Delta \omega_q = 2\chi \text{ Re } n_{w} = 2\chi \bar{n} \]

- Imaginary part: **ensemble dephasing**
  \[ \Gamma = 2\chi \text{ Im } n_{w} = \frac{8\chi^2 \bar{n}}{\kappa} \]
Nontrivial example: Hamilton-Jacobi Theory

Schrodinger Equation: \( i\hbar \partial_t |\psi(t)\rangle = \left[ \frac{\hat{p}^2}{2m} + V(\hat{x}) \right] |\psi(t)\rangle \)

Hamilton's Principle Function: \( S(t, x) \equiv -i\hbar \ln \langle x | \psi(t) \rangle \)

Momentum defined in the usual way is the weak value of the momentum operator:

\[
p_w(t, x) \equiv \partial_x S(t, x) = \frac{-i\hbar \partial_x \psi(t, x)}{\psi(t, x)} = \frac{\langle x | \hat{p} | \psi(t) \rangle}{\langle x | \psi(t) \rangle}
\]

Schrodinger's Equation is equivalent to a quantum Hamilton-Jacobi Equation:

\[
\partial_t S(t, x) + H_w[t, x, p_w(t, x)] = 0 \quad H_w[t, x, p_w(t, x)] = \frac{\langle x | \hat{p}^2/2m + V(\hat{x}) | \psi \rangle}{\langle x | \psi \rangle}
\]

Imaginary part is a continuity equation for probability. Real part is classical HJ Equation:

\[
\text{Re} H_w[t, x, p(t, x)] = \left(\text{Re} p_w(t, x)\right)^2/2m + V(x) + Q(x)
\]

"Quantum Potential": Weak Variance of momentum away from mean weak value

\[
Q(x) = \frac{\langle x | (\hat{p} - \text{Re} p_w(t, x))^2 | \psi(t) \rangle}{\langle x | \psi(t) \rangle} = -\frac{\hbar^2}{2m} \left( \partial_x^2 \langle \psi(x, t) \rangle \right)
\]
Nontrivial example: Classical field streamlines

Weak values also appear as **physical properties** of a classical field, even when there is not an obvious "weak measurement" at the level of individual field quanta.

\[
\text{Re } \mathbf{p}(\mathbf{r}) = \text{Re} \frac{\langle x | \hat{p} | \psi \rangle}{\langle x | \psi \rangle} = \frac{\omega}{c^2} S_o(\mathbf{r})
\]

Momentum weak value proportional to the local (orbital part of the) **Poynting vector** $S_o$ of an optical field, scaled by the frequency $\omega$ and energy density $W$.

This can be *measured* and used to reconstruct "averaged trajectories" for the mean momentum streamlines.

The average local momentum corresponds to the local **optical pressure** felt by small probe particles, and thus the momentum part of the **stress-energy tensor**.

Kocsis et al., Science (2011)
Bliokh et al. NJP (2013)
Nontrivial Example: Bessel beams

Both real and imaginary parts of this local momentum average also describe physical properties of classical "vortex beams" like Bessel beams.

$$p_w(r) = \frac{\langle x | \hat{p} | \psi \rangle}{\langle x | \psi \rangle}$$

The real part of the momentum weak value appears as the circulating local orbital momentum that can be transferred to probe particles by pushing them around in circular orbits.

The imaginary part is directed radially and confines the optical intensity into concentric rings, similarly trapping probe particles to orbit only within the optical rings.
Quantum Field Theory

Boundary conditions:
- Past time: $t_I$:
- Future time: $t_F$: \langle F \rangle

"Test charge" current:
- Intermediate time $t$: $J(x)$
- Take limit as $J \to 0$

But wait, there's more!

Schwinger taught us to do exactly the same thing in quantum field theory to probe mean (classical) fields and their correlations.

Schwinger, PR (1951)
QFT: Classical Mean Fields

Generating Functional:
\[ W[J] = -i\hbar \ln \langle F | \hat{U}[J] | I \rangle \]

Effective Action:
\[ \Gamma[\varphi] = W[J] - \int d^4x J(x) \varphi(x) \]

"Classical" Field:
\[ \varphi(x) \equiv \left. \frac{\delta W[J]}{\delta J(x)} \right|_{J \to 0} = \frac{\langle F(t) | \hat{\varphi}(x) | I(t) \rangle}{\langle F(t) | I(t) \rangle} \]

"Classical" Equation of Motion:
\[ \frac{\delta \Gamma[\varphi]}{\delta \varphi(x)} = -J(x) \to 0 \]

All classical fields according to QFT are weak values of the field operators.
Summary

*Weak values* are prevalent in the quantum formalism, even without performing weak measurements, which means that *Kirkwood-Dirac quasiprobabilities* are also!

- Conditioned average of weakly measured observable
- Spectral shifts due to perturbations
- Ensemble-averaged dynamical parameters
- Classical mean field properties
- Classical limit for observable values
- ...and many more examples

Thank you!