QuiDiQua : Quasiprobability Distributions in Quantum Mechanics and Quantum Information. Kirkwood-Dirac-Wigner

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# Wigner entropy and majorization in quantum phase space

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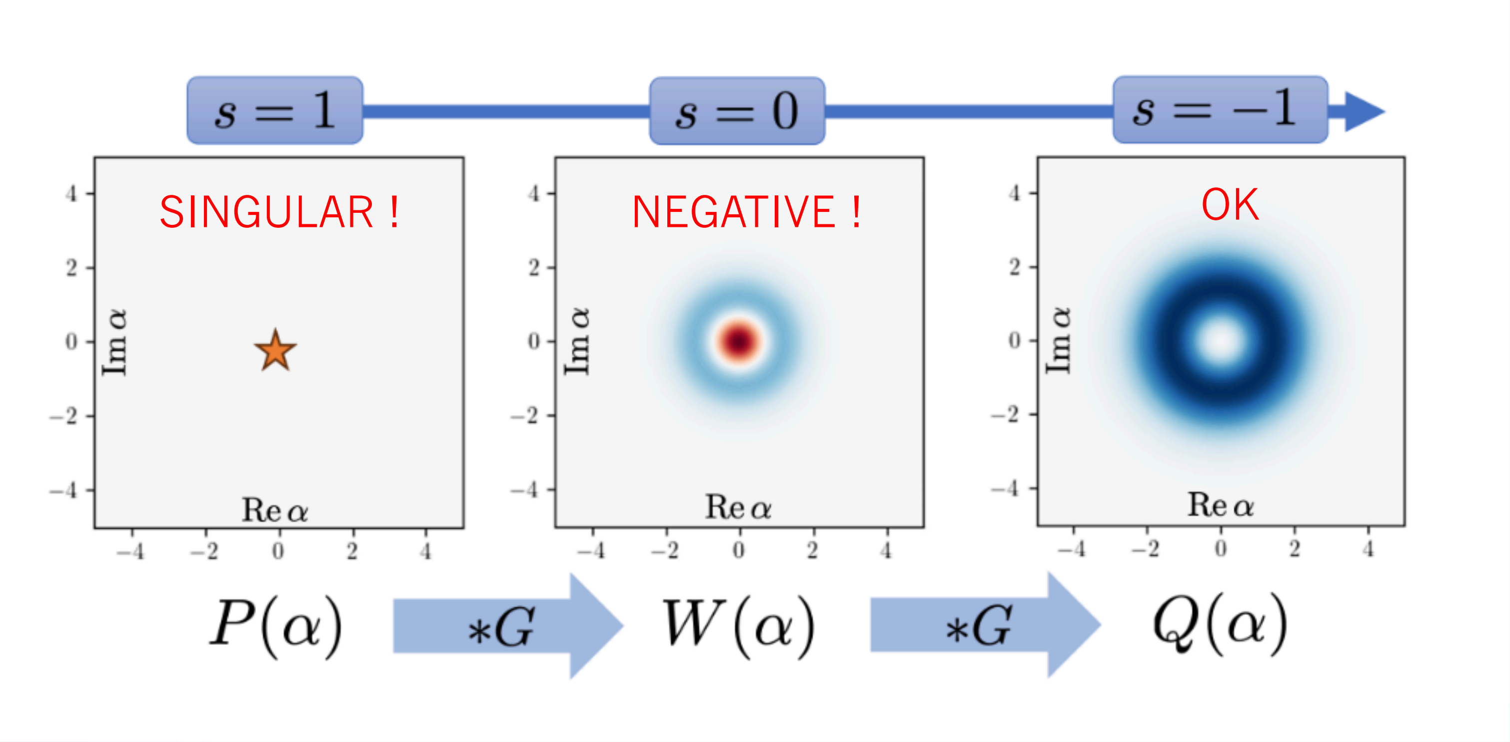
# Quasiprobabilities in phase space



- Glauber P–function $\hat{\rho} = \int P(\alpha)\,|\alpha\rangle\langle\alpha|\,\mathrm{d}\alpha$
- Wigner W–function $W(\alpha) = \frac{1}{\pi}\,\mathrm{tr}\left(\hat{D}_\alpha^\dagger \hat{\rho} \hat{D}_\alpha \,(-1)^{\hat{n}}\right)$ ALWAYS REGULAR
- Husimi Q–function $Q(\alpha) = \frac{1}{\pi}\,\langle\alpha|\hat{\rho}|\alpha\rangle$ ALWAYS NONNEGATIVE

# Measure of uncertainty in phase space ?

genuine probability density

☐ Based on Husimi fct $Q(\alpha) \geq 0$ ← ⇒ *Wehrl entropy*

but lacks some key properties:

- all Gaussian pure states are not minimum-uncertainty states
- not invariant under Gaussian unitaries
- linked to one preferred measurement (heterodyne)

regular quasiprobability density

☐ Based on Wigner fct $W(\alpha) \lessgtr 0$ ← ⇒ " *Wigner entropy* "

enjoys several key properties:

- invariant under Gaussian unitaries (area preservation)
- natural link with continuous majorization theory
- marginals of *W* are genuine probability densities

# Most natural uncertainty functional ?

Wigner entropy $$h(W) = -\int W(x,p) \ln W(x,p)\, \mathrm{d}x\, \mathrm{d}p$$

*Shannon differential entropy of Wigner function*

Two main claims :

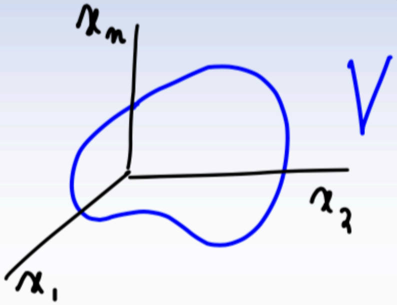
- *h* is a legitimate uncertainty functional for all positive Wigner functions (but then we have to solve some open conjectures)
- *h* remains a sound quantity when extended to negative Wigner functions (but then we have to deal with complex entropies)

Initial question : how to build entropic uncertainty relations (EUR) that are invariant under Gaussian unitaries ?

# Shannon differential (continuous) entropy

$$h(f) = -\int f(x) \ln f(x)\, dx$$

with $f(x)$ = prob. density of $x$



$V \sim e^{n\, h(x)}$

typical volume

measure of spreading of $f$

- 

$h = \ln \Delta$

- $h \gtrless 0$ .... defined up to irrelevant constant!

- $h(f) \leq h(\underline{f_G}) = \frac{1}{2} \ln(2\pi e \sigma^2)$ for fixed variance $\sigma^2$

Gaussian distribution is most random

# Entropic uncertainty relation

Bialynicki-Birula & Mycielski. (1975 )

$$h(x) + h(p) \geqslant \ln(\pi e)$$

expresses the complementarity between x- and p-quadratures

(with $\hbar = 1$) ↑ vacuum

classical

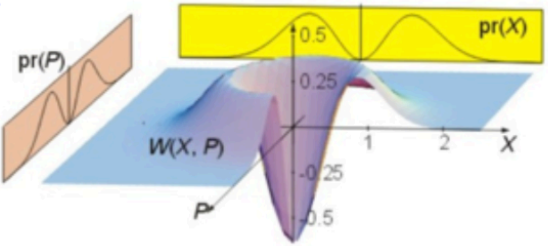
$$\begin{cases} h(x) = -\int f(x) \ln f(x) \, dx \; ; \\ h(p) = -\int g(p) \ln g(p) \, dx \; ; \end{cases}$$

$$f(x) = \mathrm{Tr}(\hat{\rho}\, \hat{x}) = \int W(x,p) \, dp$$

$$g(p) = \mathrm{Tr}(\hat{\rho}\, \hat{p}) = \int W(x,p) \, dx$$

$h(x)$ measures spreading in $x$

$h(p)$ measures spreading in $p$



# … invariant under Gaussian unitaries ?

Heisenberg, Kennard

$$\sigma_x^2\,\sigma_p^2 \geq \frac{1}{4}$$

$$\sigma_x^2 = \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2$$

$$\sigma_p^2 = \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2$$

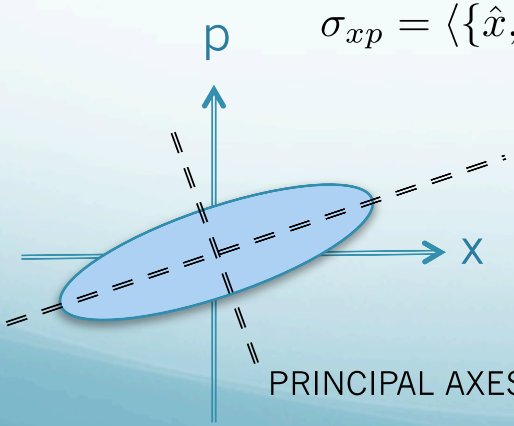
Bialynicki-Birula & Mycielski

$$h(x) + h(p) \geq \ln(\pi e)$$

Schrödinger, Robertson

$$\det \gamma \geq \frac{1}{4} \qquad \gamma = \begin{pmatrix} \sigma_x^2 & \sigma_{xp} \\ \sigma_{xp} & \sigma_p^2 \end{pmatrix}$$

$$\sigma_{xp} = \langle \{\hat{x},\hat{p}\}/2 \rangle - \langle \hat{x} \rangle \langle \hat{p} \rangle$$



Seeking EUR saturated
for all pure Gaussian states ?

Several attempts :

- improved (hybrid) inequality
- multicopy inequality
- Wigner entropy

$$h(W) \geq \ln(\pi e)$$

*sort of joint entropy* $h(x,p)$

# Entropy power

Key notion in real-valued signal info. th.

Remember entropy of Gaussian dist. $h(f_G) = \frac{1}{2} \ln (2\pi e \sigma^2)$ of variance $\sigma^2$

$$\longleftrightarrow \quad \sigma^2 = \frac{1}{2\pi e} e^{2h(f_G)}$$

<u>Def</u> : $N_f = \frac{1}{2\pi e} e^{2h(f)}$ <u>entropy power</u> of arbitrary distribution $f$

$\equiv$ variance of the Gaussian distribution $f_G$ that has the same entropy as $f$

<u>Property</u> : $N_f \leqslant \sigma^2$

"entropy power" "true power"

# Entropy-power uncertainty relation

… back to entropic uncertainty relation

$$h(x) + h(p) \geqslant \ln(\pi e) \qquad (\text{with } \hbar = 1)$$

since $\begin{cases} N_x = \frac{1}{2\pi e}\, e^{2h(x)} \\ N_p = \frac{1}{2\pi e}\, e^{2h(p)} \end{cases}$ ← entropy powers

$$\Leftrightarrow \quad \boxed{N_x \cdot N_p \geqslant \frac{1}{4}}$$

ENTROPY-POWER UNCERTAINTY RELATION

It implies Heisenberg relation

since $\begin{cases} N_x \leqslant \sigma_x^2 \\ N_p \leqslant \sigma_p^2 \end{cases}$

$$\boxed{\sigma_x^2 \cdot \sigma_p^2 \geqslant N_x \cdot N_p \geqslant \frac{1}{4}}$$

↑ vacuum

# Improved uncertainty relation

(hybrid form)

Schrödinger-Robertson uncertainty rel.

$|\gamma| \geqslant \frac{1}{4}$ with $\gamma = \begin{pmatrix} \sigma_x^2 & c \\ c & \sigma_p^2 \end{pmatrix}$

IMPROVEMENT
OVER
HEISENBERG

... can be written formally as $\sigma_x^2 \sigma_p^2 \geqslant \underbrace{\frac{\sigma_x^2 \cdot \sigma_p^2}{|\gamma|}}_{\geqslant 1} \cdot \frac{1}{4}$

Hadamard ineq.

Tighter entropy-power uncertainty rel.

$$\boxed{N_x \cdot N_p \overset{?}{\geqslant} \frac{\sigma_x^2 \cdot \sigma_p^2}{|\gamma|} \cdot \frac{1}{4}}$$

Conjecture (it is proven that Gaussian pure states are local minimizers)

which implies Schrödinger-Robertson uncertainty rel. $|\gamma| \geqslant 1/4$

# Wigner-entropy uncertainty relation

$$\boxed{h(x,p) \overset{?}{\geqslant} \ln(\pi e)}$$

Wigner entropy

$$-\int W(x,p) \ln W(x,p)\, dx\, dp$$

👍 Saturated for all Gaussian pure states

$$h_G = \frac{1}{2} \ln\Big( (2\pi e)^2 \underbrace{|\gamma|}_{= 1/4} \Big)$$
$$= \ln(\pi e)$$

👍 Invariant under any Gaussian unitary transformation

$$\begin{pmatrix} x' \\ p' \end{pmatrix} = S \begin{pmatrix} x \\ p \end{pmatrix} + \begin{pmatrix} x_0 \\ p_0 \end{pmatrix}$$

symplectic matrix

$$\Rightarrow h(x',p') = h(x,p) + \ln \underbrace{|S|}_{1}$$
$$= h(x,p)$$

👍 Wigner entropy is concave

👎 Only defined for states with $W(x,p)$ positive everywhere

# Wigner entropy conjecture

$$\boxed{h(x,p) \equiv h(W) \overset{?}{\geq} \ln(\pi e) \qquad \forall W \geq 0}$$

Tighter bound than Bialynicki-Birula & Mycielski EUR since

$$h(x,p) = h(x) + h(p) - \underbrace{I(x:p)}_{\geq 0} \overset{?}{\geq} \ln(\pi e)$$

subadditivity of
Shannon entropy

Set of positive Wigner functions

$$\mathcal{W}_+ = \{W(x,p) : W(x,p) \geq 0, \forall (x,p)\}$$ is convex

MINIMIZING A CONCAVE FUNCTION $h(W)$ OVER A CONVEX SET $\mathcal{W}_+$

→ SHOULD BE PROVEN FOR EXTREMAL STATES

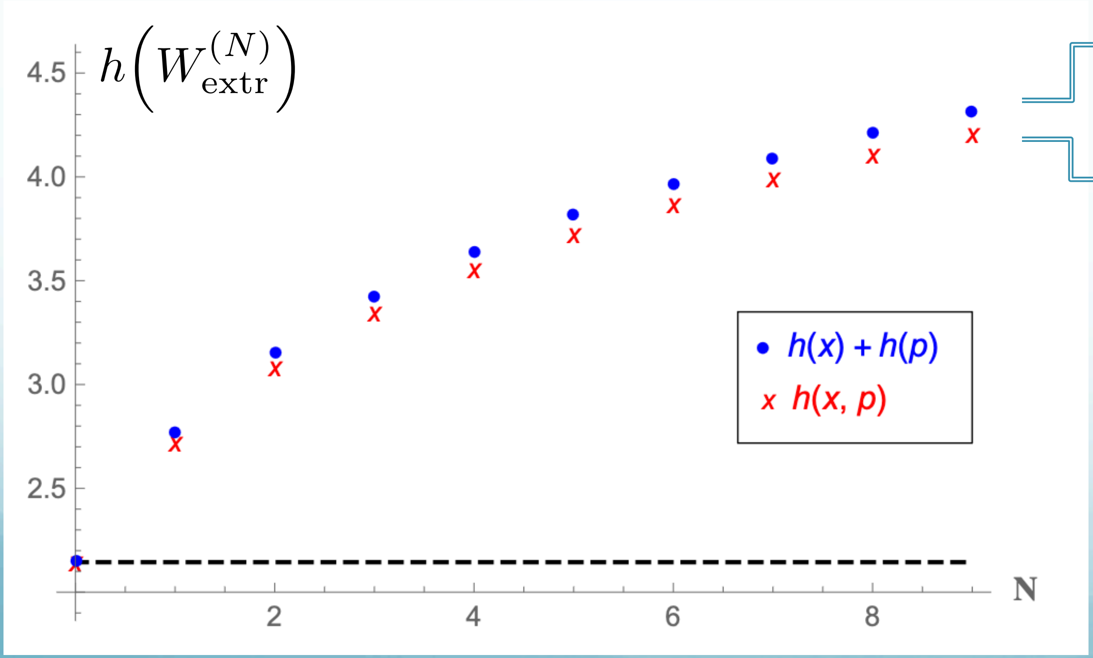
Proof for passive states $\hat{\rho}_{\text{pass}} = \sum_k w_k \, |k\rangle\langle k|$ with $w_k \geq w_{k+1}$

*(monotonically decreasing weights)*

$$\boxed{W_{\text{pass}} \in \mathcal{W}_+}$$

$$= \sum_{N=0}^{\infty} p_N \, \hat{\rho}_{\text{extr}}^{(N)}$$ with $p_N \geq 0, \quad \Sigma_N p_N = 1$

$$\hat{\rho}_{\text{extr}}^{(N)} = \frac{1}{N+1} \sum_{n=0}^{N} |n\rangle\langle n|$$



Bialynicki-Birula & Mycielski EUR

Wigner EUR

Proof for passive states $\hat{\rho}_{\text{pass}} = \sum_k w_k \, |k\rangle\langle k|$ with $w_k \geq w_{k+1}$

*(monotonically decreasing weights)*

$$h\left(\sum_k w_k \, W_{|k\rangle}\right) = h\left(\sum_N p_N \, W_{\text{extr}}^{(N)}\right)$$

$$\geq \sum_N p_N \, h\left(W_{\text{extr}}^{(N)}\right)$$ concavity of Wigner entropy *h*

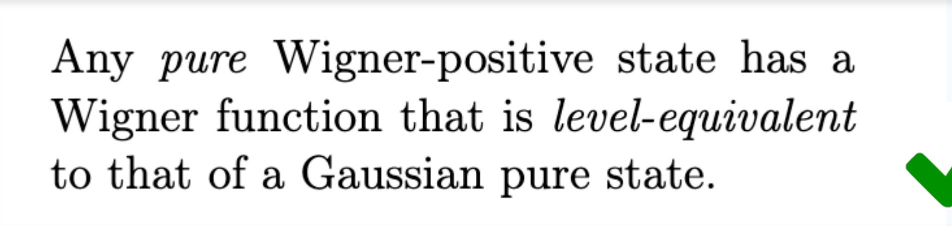
$$= \sum_N p_N \, h\left(\frac{1}{N+1}\sum_{i=0}^{N} W_{|i\rangle}\right)$$

$$\geq \sum_N p_N \, \frac{1}{N+1}\sum_{i=0}^{N}\left(h(x)_{|i\rangle} + h(p)_{|i\rangle}\right)$$ Van Herstraeten & NJC

$$= \sum_k w_k \, \underbrace{\left(h(x)_{|k\rangle} + h(p)_{|k\rangle}\right)}_{\geq \ln(\pi e)}$$ Bialynicki-Birula & Mycielski

(only valid provided $w_k \geq w_{k+1}$ )

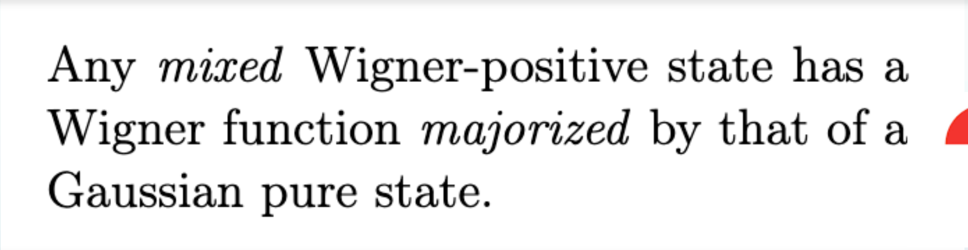
# Majorization relation in phase space ?



Any *pure* Wigner-positive state has a Wigner function that is *level-equivalent* to that of a Gaussian pure state. ✓

$$W(x,p) \equiv W_0(x,p)$$
$$\forall \text{ pure } W \geq 0$$

According to Hudson theorem, these are all Gaussian pure states which are convertible into each other by Gaussian unitaries, hence they have the same “level function” (they are “level–equivalent”)



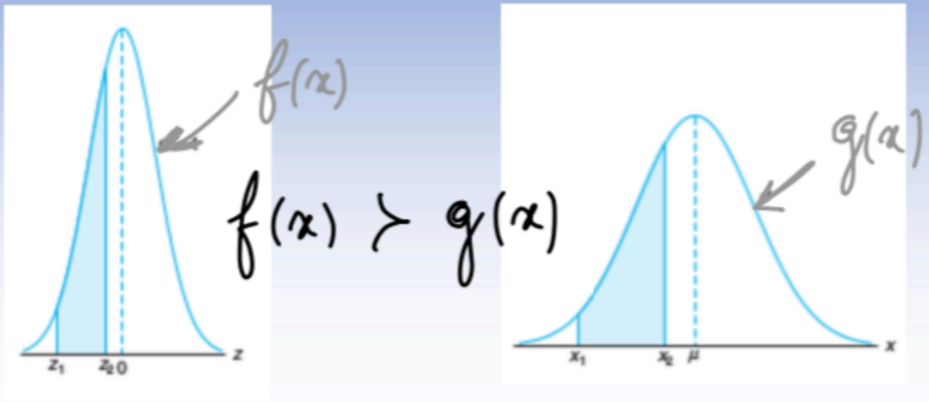
Any *mixed* Wigner-positive state has a Wigner function *majorized* by that of a Gaussian pure state. ?

$$W(x,p) \overset{?}{\prec} W_0(x,p)$$
$$\forall W \geq 0$$

Any Wigner–positive state is more disordered (in phase space) than Gaussian pure states

# Continuous majorization relations

pre-order relation between probability densities



$f(x) \succ g(x)$

Need to define decreasing rearrangement of function

$f(x) \Rightarrow f^{\downarrow}(x)$

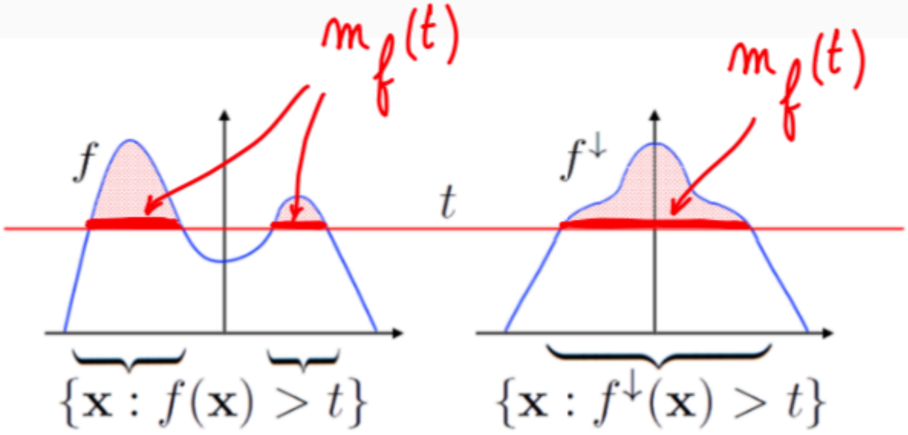
- $m_f(t) = \mathrm{Vol}\left(\{x : f(x) > t\}\right)$

Lebesgue measure

level function

indicator function

- $f^{\downarrow}(x) = \int_0^{\infty} dt \; I_{x \in \text{"ball" of volume } m_f(t)}$

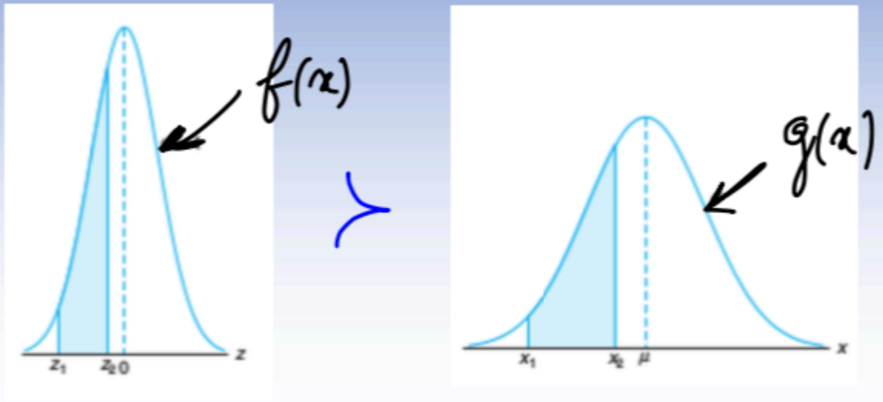


symmetric (even) and monotonically decreasing for $x \geq 0$

# Continuous majorization relations

$f(x) \succ g(x)$

f is more ordered than g



$$\Leftrightarrow \int_{B(0,r)} f^{\downarrow}(x)\,dx \geqslant \int_{B(0,r)} g^{\downarrow}(x)\,dx \qquad \forall r \geqslant 0$$

cumulative function

→ ball of radius r (centered on 0)

with $f^{\downarrow}$ and $g^{\downarrow}$ decreasing rearranged functions

$$\Leftrightarrow \int_{-\infty}^{+\infty} \phi(f(x)) \geqslant \int_{-\infty}^{+\infty} \phi(g(x)) \qquad \forall \text{ convex function } \phi : \mathbb{R}^{+} \to \mathbb{R}$$

- Link with differential entropy : $\phi(x) = -x \log x \rightarrow$ concave

$$f \succ g \quad \Rightarrow \quad \underbrace{-\int f(x) \ln f(x)\, dx}_{h(f)} \leq \underbrace{-\int g(x) \ln g(x)\, dx}_{h(g)}$$

$$\not\Leftarrow$$

... entropy can only increase with disorder

- Majorization relation invariant under decreasing rearrangement

$$f \succ g \quad \Leftrightarrow \quad f^{\downarrow} \succ g^{\downarrow}$$ (preserving level function $m_f$)

- Equivalence : $\left.\begin{array}{l} f \succ g \\ g \succ f \end{array}\right\} \; f \equiv g$ (same level function $m_f = m_g$)

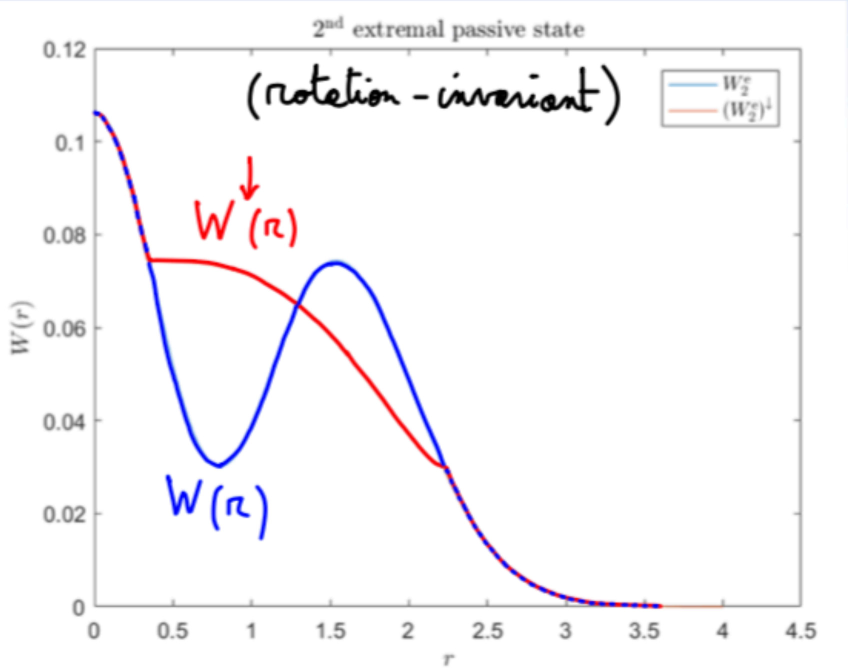
# Application to Wigner functions

with $W(x,p) \geq 0$ $\forall x,p$

- Decreasing rearranged Wigner function

$$\rho = \tfrac{1}{3}|0\rangle\langle 0| + \tfrac{1}{3}|1\rangle\langle 1| + \tfrac{1}{3}|2\rangle\langle 2|$$

radial dependence



- Majorization criterion

$$W_1 \succ W_2 \iff \int_0^R W_1^\downarrow(r)\, 2\pi r\, dr \geq \int_0^R W_2^\downarrow(r)\, 2\pi r\, dr, \quad \forall R$$

# Wigner majorization conjecture

$$W(x,p) \overset{?}{\prec} W_0(x,p) \qquad \forall W \geq 0$$

◆ All positive Wigner functions *W are "majorized"* by the Wigner function *$W_0$* of the vacuum state (they are more disordered in the sense of majorization theory)

◆ All positive Wigner functions *W* that are related by a Gaussian unitary are "*equivalent*" (in the sense of majorization theory)

◆ This implies Wigner entropy conjecture

$$h(W) \geq h(W_0) = \ln(\pi e) \qquad \forall W \geq 0$$

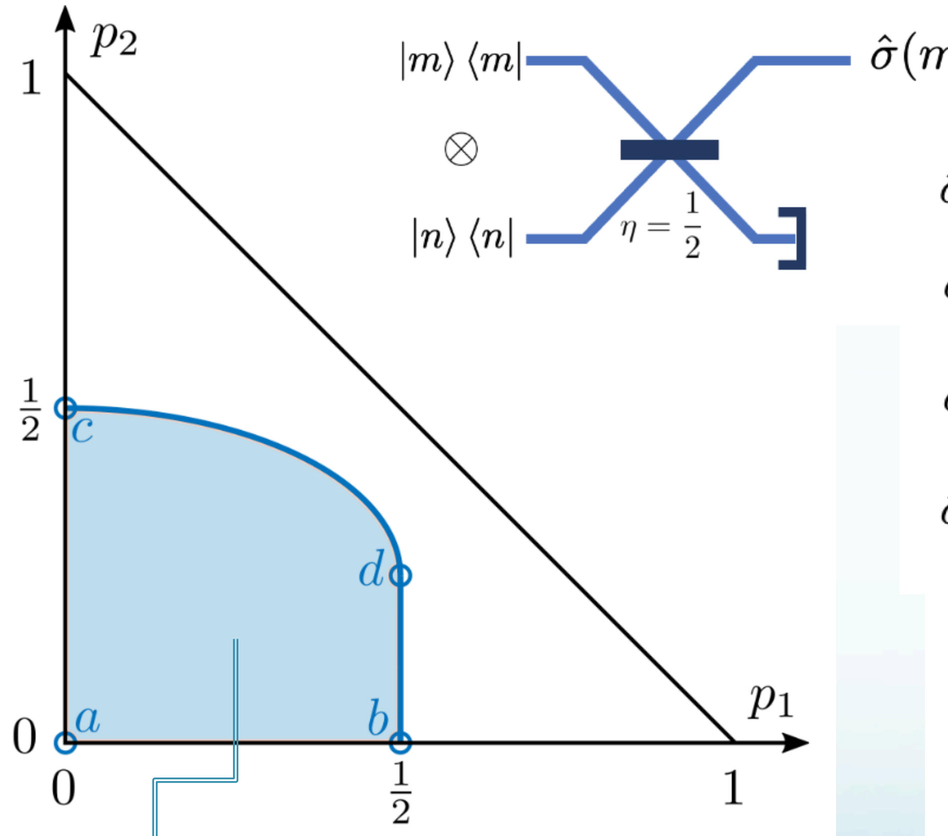
◆ Also implies a similar conjecture on all Wigner–Rényi entropies

$$h_\alpha(W) \geq h_\alpha(W_0) \qquad \forall W \geq 0$$

(same for all Shur-concave functionals)

# Proof for Wigner-positive phase-invariant states restricted to ≤ 2 photons

$$\hat{\rho} = (1 - p_1 - p_2)\,|0\rangle\langle 0| + p_1\,|1\rangle\langle 1| + p_2\,|2\rangle\langle 2|$$



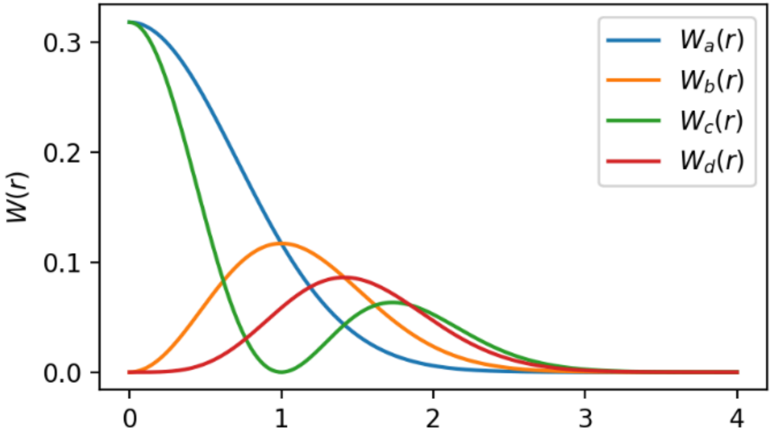
Wigner-positive set

$$\hat{\sigma}_a \equiv \hat{\sigma}(0,0) = |0\rangle\langle 0|$$

$$\hat{\sigma}_b \equiv \hat{\sigma}(1,0) = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

$$\hat{\sigma}_c \equiv \hat{\sigma}(1,1) = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|2\rangle\langle 2|$$

$$\hat{\sigma}_d \equiv \hat{\sigma}(2,0) = \frac{1}{4}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| + \frac{1}{4}|2\rangle\langle 2|$$



$$\left[\begin{aligned} W_a(r) &= W_0(r) = \frac{1}{\pi}\exp\left(-r^2\right), \\ W_b(r) &= \frac{1}{\pi}\exp\left(-r^2\right) r^2, \\ W_c(r) &= \frac{1}{\pi}\exp\left(-r^2\right)\left(r^2-1\right)^2, \\ W_d(r) &= \frac{1}{\pi}\exp\left(-r^2\right)\frac{1}{2}r^4. \\ V_t(r) &= \frac{t+1}{2\pi}\exp\left(-r^2\right)\left(r^2-1+\sqrt{\frac{1-t}{1+t}}\right)^2 \end{aligned}\right. \quad \Longrightarrow \quad \left[\begin{aligned} f_0(x) &= \exp\left(-x\right), \\ f_b(x) &= \exp\left(-x\right) x, \\ f_c(x) &= \exp\left(-x\right)\left(x-1\right)^2, \\ f_d(x) &= \exp\left(-x\right)\frac{1}{2}x^2, \\ g_t(x) &= \exp\left(-x\right)\frac{1}{2}(t+1)\left(x-1+\sqrt{\frac{1-t}{1+t}}\right)^2 \end{aligned}\right.$$

$t \in [0,1]$

EXAMPLE PROOF FOR $\hat{\sigma}_b = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$

$$f_b(x) = \int_0^\infty f_0^{(z)}(x)\ k(z)\,\mathrm{d}z \quad \Longrightarrow \quad f_b(x) \prec f_0(x)$$

mixture of level-equivalent functions

$$f_0^{(z)}(x) = f_0(x-z)\,\Theta(x-z) \quad \equiv f_0(x) \quad \forall z$$

with weight $k(z) = \exp(-z)$

# Dealing with Wigner–negative states ?

Tentative definitions of a (real–valued) uncertainty functional $\Phi(W)$

Desirable properties

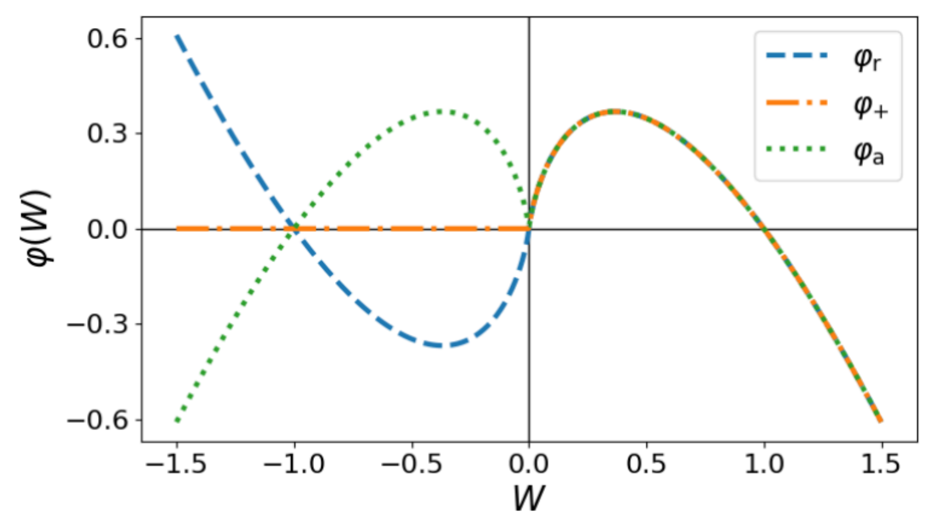
1. $\forall W \in \mathcal{W}_+ : \quad \Phi(W) = h(W)$
2. $\forall W \in \mathcal{W} : \quad \Phi(W)$ is a symmetric functional
3. $\forall W \in \mathcal{W} : \quad \Phi(W)$ is a concave functional
4. $\forall W \in \mathcal{W} : \quad \Phi(W) \leq h(\rho_x) + h(\rho_p)$
5. $\forall W \in \mathcal{W} : \quad \ln \pi + 1 \leq \Phi(W).$

Examples : $\Phi(W) = \int \phi\big(W(x,p)\big)\, \mathrm{d}x\, \mathrm{d}p$

with $\phi(x) = -x \ln |x|$

or $\phi(x) = -|x| \ln |x|$

or $\phi(x) = -\Theta(x)\, x \ln x$



... satisfies prop. 1 and 2
but cannot be concave

# Complex–valued Wigner entropy

The fact that $\Phi(W)$ should coincide with Wigner entropy over the full set of positive Wigner functions (i.e., for $W \in \mathbb{R}^+$) suggests to take the analytic continuation of Shannon entropy in the complex plane

$$\begin{aligned} h_c(W) &= -\int W(x,p)\, \underline{\underline{\ln W(x,p)}}\, \mathrm{d}x\, \mathrm{d}p \\ &= h_r(W) + i\, h_i(W) \end{aligned}$$

complex logarithm function

REAL PART

$$h_r(W) = -\int W(x,p)\, \underline{\underline{\ln |W(x,p)|}}\, \mathrm{d}x\, \mathrm{d}p$$

IMAGINARY PART

$$h_i(W) = -\int W(x,p)\, \underline{\underline{\arg W(x,p)}}\, \mathrm{d}x\, \mathrm{d}p$$

# Imaginary part of complex Wigner entropy

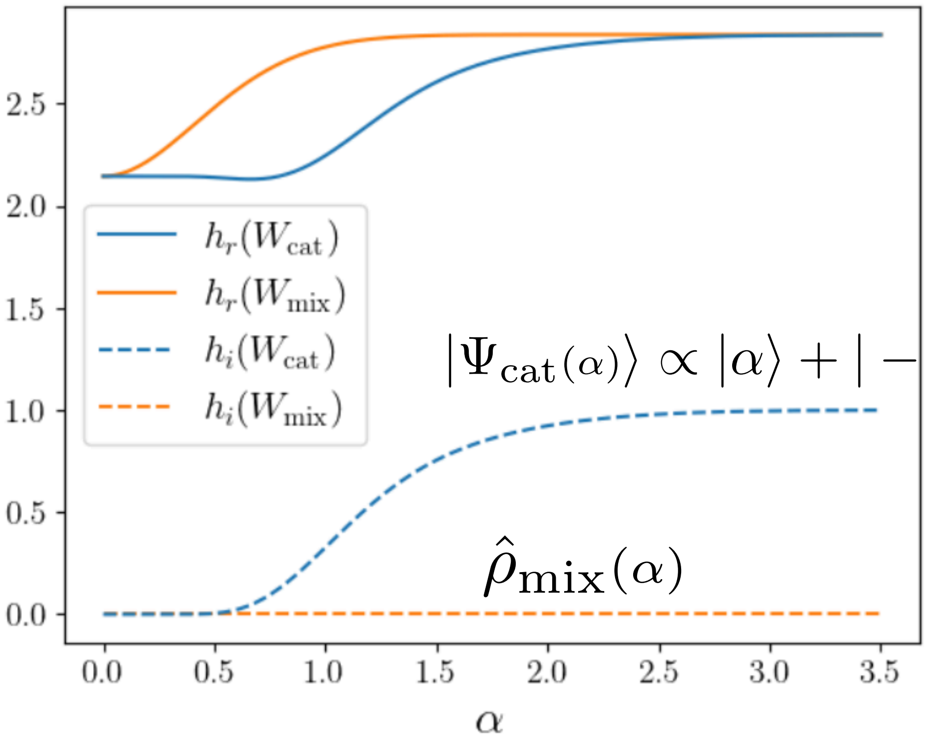
$$h_i(W) = \pi\,\mathrm{Vol}_-(W) = -\pi \int_{\mathcal{D}} W(x,p)\,\mathrm{d}x\,\mathrm{d}p \;\; \geq 0$$

Wigner negative domain

$$\mathrm{Vol}_-(W) = -\int_{W(x,p)<0} W(x,p)\,\mathrm{d}x\,\mathrm{d}p$$

negative Wigner volume

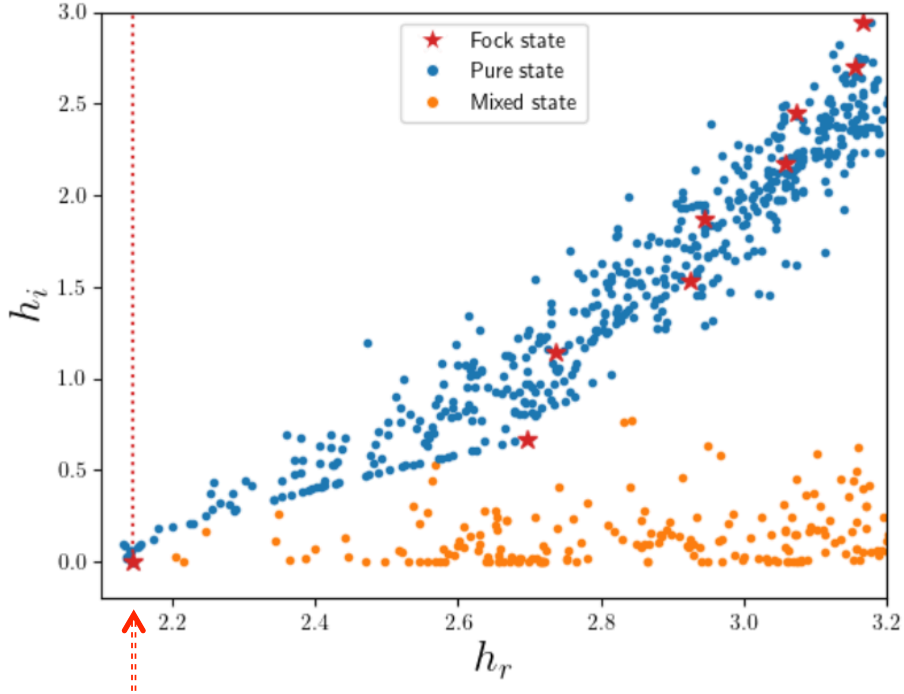
(measure of non-classicality,
invariant under Gaussian unitaries)



$|\Psi_{\mathrm{cat}(\alpha)}\rangle \propto |\alpha\rangle + |-\alpha\rangle$ cat state

$+\ i$

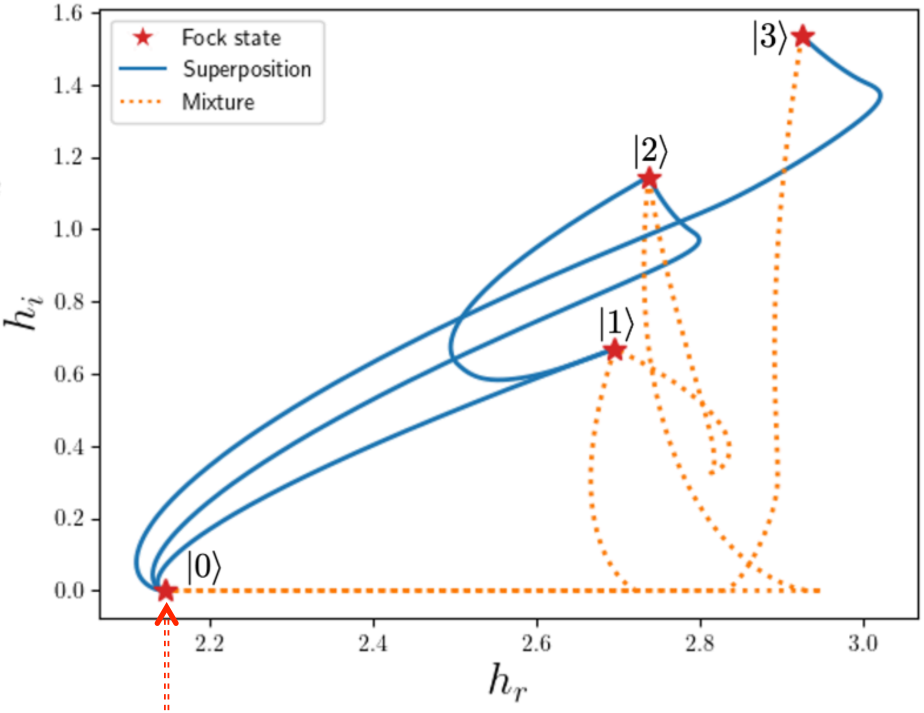
$\hat{\rho}_{\mathrm{mix}}(\alpha)$ decohered cat state (mixture)



$\ln(\pi e)$

Complex uncertainty relation ?

Physically allowed domain
in complex entropy plane ?



$\ln(\pi e)$

*Property* : Both the real and imaginary parts of the complex Wigner entropy are symmetric functionals, hence invariant under symplectic transformations

$$h_r(W) = \int \varphi_r\left(W(x,p)\right)\,\mathrm{d}x\,\mathrm{d}p,$$

with $\varphi_r(x) = -x\ln|x|$

$$h_i(W) = \int \varphi_i\left(W(x,p)\right)\,\mathrm{d}x\,\mathrm{d}p,$$

with $\varphi_i(x) = -x\arg(x)$

*Property*: Both the real and imaginary parts of the complex Wigner entropy evolve (under Gaussian additive noise) according to an extended *de Bruijn identity*

$$\frac{\mathrm{d}}{\mathrm{d}t}h_c(W) = \frac{1}{2}J_c(W).$$

complex-valued Wigner entropy

complex-valued Fisher information

# Complex-valued Fisher information

... analytic continuation of Fisher information in the complex plane

$$J_c(W) = \int \left( \frac{\mathrm{d}W}{\mathrm{d}x} \frac{\mathrm{d}(\ln W)}{\mathrm{d}x} + \frac{\mathrm{d}W}{\mathrm{d}p} \frac{\mathrm{d}(\ln W)}{\mathrm{d}p} \right) \mathrm{d}x\, \mathrm{d}p$$

$$= \int \boldsymbol{\nabla} W \cdot \boldsymbol{\nabla} \underline{\ln W}\ \mathrm{d}x\, \mathrm{d}p$$

$$= J_r(W) + i\, J_i(W),$$

complex logarithm function

REAL PART

$$J_r(W) = \int \boldsymbol{\nabla} W \cdot \underline{\underline{\boldsymbol{\nabla} \ln |W|}}\ \mathrm{d}x\, \mathrm{d}p.$$

$$= \int W^{-1}(x,p)\, \|\boldsymbol{\nabla} W\|^2\ \mathrm{d}x\, \mathrm{d}p,$$

$$\geq 0$$

IMAGINARY PART

$$J_i(W) = \int \boldsymbol{\nabla} W \cdot \underline{\underline{\boldsymbol{\nabla} \arg W}}\ \mathrm{d}x\, \mathrm{d}p,$$

$$= -\pi \int_{\mathcal{D}} \Delta W\ \mathrm{d}x\, \mathrm{d}p,$$

$$= -\pi \oint_{\partial D} \boldsymbol{\nabla} W \cdot \boldsymbol{n}\, \mathrm{d}s.$$

$$\leq 0$$

# (complex) de Bruijn identity

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} h_c(W) = \frac{1}{2} J_c(W).}$$

when W undergoes Gaussian diffusion process $$\frac{\mathrm{d}}{\mathrm{d}t} W = \frac{1}{2} \Delta W,$$

$\hat{\rho}(0)$

GAUSSIAN ADDITIVE-NOISE CHANNEL

$\hat{\rho}(t)$

$$\hat{x}(t) = \hat{x}(0) + \sqrt{t}\ \hat{x}_{\mathrm{vac}}$$

$$\hat{p}(t) = \hat{p}(0) + \sqrt{t}\ \hat{p}_{\mathrm{vac}}$$

# Conclusions

REAL-VALUED $$h(W) = -\int W(x,p) \ln W(x,p) \, \mathrm{d}x \, \mathrm{d}p \qquad \forall W \geq 0$$

COMPLEX-VALUED $$h_c(W) = -\int W(x,p) \ln W(x,p) \, \mathrm{d}x \, \mathrm{d}p \qquad \forall W \lessgtr 0$$

- Natural symplectic–invariant uncertainty measure in phase space
- Yields tighter EUR but limited to $\mathcal{W}^+$
- Full proof of Wigner entropy conjecture is needed
- Full proof of Wigner majorization conjecture is needed

- Complex extension seems reasonable  
  (c-numbers are ubiquitous in quantum mechanics)
- The usual playground for quasiprobability distributions  
  in quantum phase space is precisely the complex plane
- Yet, the operational meaning of complex Wigner entropy is missing  
  (in terms of – possibly complex – typical volume)

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